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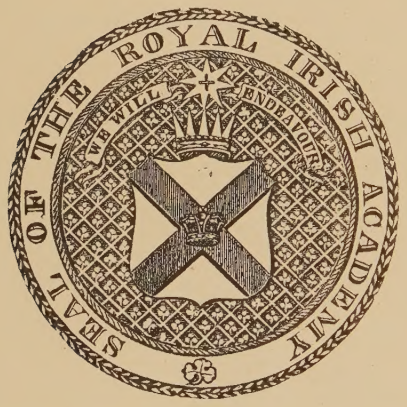

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CONTENTS.

PAGE

ALEXANDER, T. and THOMSON, A. W. :—

On Two-Nosed Catenaries and their application to the Design of Segmental Arches. (*Plate III.*), ... 43

[Published November, 1888.]

On Elliptographs, and on a Mechanical Rocker for Detecting Oscillations. To which is added A Mathematical Investigation of the Free Period of the Rocker by TARLETON, F. A., ... 673

[Published March, 1892.

BALL, SIR R. S. :—

On the Plane Sections of the Cylindroid; being the Seventh Memoir on the Theory of Screws. (*Plates I. and II.*), ... 1

[Published December, 1887.]

On the Theory of the Content, ... 123

[Published January, 1889.]

Eighth Memoir on the Theory of Screws, showing how Plane Geometry illustrates General Problems in the Dynamics of a Rigid Body with Three Degrees of Freedom. (*Plate VI.*), ...

[Published June, 1889.]

The Theory of Permanent Screws, being the Ninth Memoir on the Theory of Screws, ... 613

[Published November, 1891.]

BENNETT, E. H. :— PAGE

[*See* CUNNINGHAM, D. J.]

BERNARD, JOHN HENRY :—

On some Fragments of an Uncial MS. of Saint Cyril of Alexandria, written on Papyrus. (*Plates IX., X., XI., and XII.*), ... 653

[Published February, 1892.]

CUNNINGHAM, D. J. and BENNETT, E. H. :—

On the Brain and Eyeball of a Human Cyclopian Monster. (*Plates IV. and V.*), ... 101

[Published November, 1888.]

CUNNINGHAM, D. J. :—

On the Skeleton of the Irish Giant, Cornelius Magrath. (*Plates VII. and VIII.*), ... 553

[Published June, 1891.]

DOBERCK, W. :—

Markree Observations of Double Stars, ... 379

[Published March, 1890.]

GRAVES, CHARLES :—

On the Ogam Monument at Kilcolman, ... 33

[Published December, 1887.]

On the Focal Circles of Spherical Conics, ... 285

[Published July, 1889.]

HAUGHTON, S. :—

Geometrical Illustrations of Newlands' and Mendelejeff's Periodic Law of the Atomic Weights of the Chemical, ... 207

[Published June, 1889.]

MAC CARTHY, BARTHOLOMEW :—

The Tripartite Life of Saint Patrick: New Textual Studies, ... 183

[Published May, 1889.]

M'CAY, W. S. :— PAGE

On Three Similar Figures, with an Extension of Feuerbach's Theorem, 303

[Published July, 1889.]

PRESTON, THOMAS :—

On the Motion of a Particle, and the Equilibrium of Flexible Strings on a Spherical Surface, ... 321

[Published July, 1889.]

SOLLAS, W. J. :—

Contributions to a Knowledge of the Granites of Leinster, ... 427

[Published January, 1891.]

TARLETON, F. A. :—

[*See* ALEXANDER, T.]

THOMSON, A. W. :—

[*See* ALEXANDER, T.]

WILLIAMSON, BENJAMIN :—

On Curvilinear Coordinates, ... 515

[Published March, 1891.]

WINDLE, BERTRAM :—

On the Pectoral Group of Muscles, ... 345

[Published November, 1889.]

*** The date of the reading will be found prefixed to each Paper. The date of publication, as inserted in the above Table of Contents of this Volume, is that when the Paper was laid in a printed form on the Table of the Academy; or, when the Academy was not in Session, the date when it was on sale at the Academy's Booksellers.

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TRANSACTIONS

OF THE

ROYAL IRISH ACADEMY.

I.

ON THE PLANE SECTIONS OF THE CYLINDROID. BEING THE SEVENTH MEMOIR ON THE THEORY OF SCREWS.

By SIR R. S. BALL, LL.D., F.R.S., Royal Astronomer of Ireland. Plates I. and II.

[Read May 9, 1887.]

IN the progress of an attempt to develop, with some degree of completeness, the more fundamental parts of the Theory of Screws, I have had frequent occasion to study the plane sections of the cylindroid. The geometrical theory which is involved seems to be of sufficient interest to justify me in bringing some account of these investigations before the Royal Irish Academy.

The reader of this Paper is presumed to be familiar with the earlier portions of the Theory of Screws, which have been from time to time laid before this Academy, or published in a more complete form* elsewhere.

The conoidal cubic surface, known as the cylindroid, is defined by the equation

$$z(x^2 + y^2) = 2mxy.$$

It has a nodal line, and also contains three straight lines at infinity. A

* See *Theory of Screws:* Dublin, 1876.

plane section of the cylindroid is a cubic curve with a real or imaginary double point. Besides the general case, which will first be examined, we have three special sections to consider, in which the plane either—(1) passes through the centre, or (2) is parallel to the nodal line, or (3) touches the surface.

§ 1. **The General Case.**—Each generator of the cylindroid is the abode of a certain screw, and accordingly each point in a plane section will lie on one screw, and generally on only one. We may, accordingly, regard the several points of the cubic as in correspondence with the several screws on the cylindroid. It will often be convenient to speak of the points on the section as synonymous with the screws themselves which pass through those points.

We must first investigate the equation to the cubic curve produced by cutting the cylindroid by a plane situated in any arbitrary position.

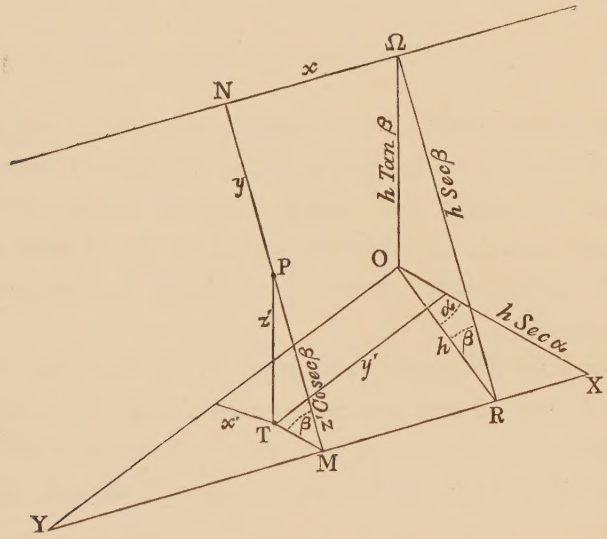


Fig. 1.

Let OX and OY (fig. 1) be the two principal screws of the cylindroid of which $O\Omega$ is the nodal line. Let $XY\Omega$ be the arbitrary plane of section. The position of this plane is defined by the magnitudes h, α, β, whereof h

is the length of the perpendicular from O on XY, α is the angle between OR and OX, and β is the angle $OR\Omega$, or the inclination of the plane of section to the principal plane of the cylindroid.

Draw through Ω the line ΩN parallel to XY; then we shall adopt ΩN as the new axis of x and ΩR as the new axis of y, so that if P be any point on the surface, we have $PN = y$ and $\Omega N = x$. The dotted letters, x', y', z' refer to the original axes of the cylindroid. Let fall PT perpendicular on the plane of OXY, and TM perpendicular to XY. Then we have $MN = R\Omega$,

whence, $$y + z' \operatorname{cosec} \beta = h \sec \beta, \qquad \text{(i.)}$$

while, if θ be the angle XOT, and OT be denoted by r, we have (fig. 2),

$$x = r \sin (\theta - \alpha),$$

or, $$x = y' \cos \alpha - x' \sin \alpha;$$

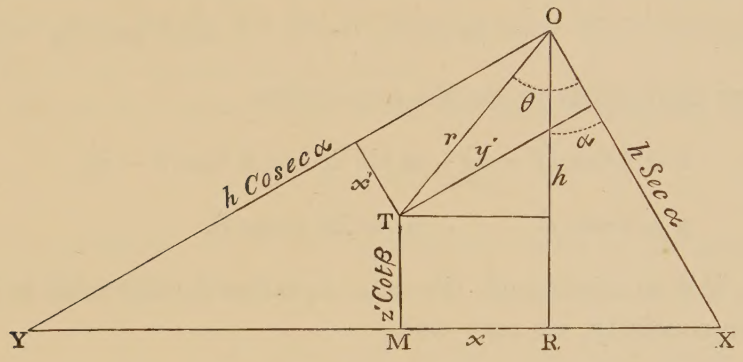


Fig. 2.

but, from obvious geometry,

$$OR = r \cos (\theta - \alpha) + MT; \qquad \text{(ii.)}$$

whence,

$$h = x' \cos \alpha + y' \sin \alpha + z' \cot \beta. \qquad \text{(iii.)}$$

Solving the equations i., ii., iii., we obtain,

$$x' = - x \sin \alpha + y \cos \beta \cos \alpha,$$

$$y' = + x \cos \alpha + y \cos \beta \sin \alpha,$$

$$z' = \quad h \tan \beta - y \sin \beta.$$

It appears from these that

$$x'^2 + y'^2 = x^2 + y^2 \cos^2\beta,$$

$$x'y' = xy \cos\beta \cos 2\alpha + (y^2 \cos^2\beta - x^2) \sin\alpha \cos\alpha.$$

The equation of the cylindroid gives

$$z'(x'^2 + y'^2) = 2mx'y';$$

whence we deduce, as the required equation of the section,

$$(h\tan\beta - y\sin\beta)(x^2 + y^2\cos^2\beta) = 2mxy\cos\beta\cos 2\alpha + 2m\sin\alpha\cos\alpha(y^2\cos^2\beta - x^2);$$

or, arranging the terms,

$$\sin\beta\cos^2\beta y^3 + \sin\beta y x^2 - (m\sin 2\alpha + h\tan\beta)x^2 + 2mxy\cos\beta\cos 2\alpha$$
$$+ (m\sin 2\alpha\cos^2\beta - h\sin\beta\cos\beta)y^2 = 0.$$

It is often convenient to use the expressions

$$x = h\tan(\theta - \alpha) - m\sin 2\theta\cot\beta\tan(\theta - \alpha),$$

$$y = h\sec\beta \qquad - m\sin 2\theta\operatorname{cosec}\beta,$$

from which, if θ be eliminated, the same equation for the cubic is obtained; or still, more concisely, we may write

$$x = y\cos\beta\tan(\theta - \alpha),$$

$$y = h\sec\beta - m\sin 2\theta\operatorname{cosec}\beta.$$

This cubic has one real asymptote, the equation of which is

$$y\sin\beta = m\sin 2\alpha + h\tan\beta,$$

and the asymptote cuts the curve in the finite point of which the x is

$$-\tan 2\alpha(h + m\sin 2\alpha\cot\beta).$$

The value of θ at this point is $-\alpha$.

In Plate I. will be found a drawing of this curve. The following are the values of the constants adopted:—

$$\alpha = 25^\circ; \quad \beta = 26^\circ; \quad h = 18; \quad m = 28{\cdot}9;$$

with which the equations become,

$$x = {\cdot}9y \tan(\theta - 25),$$

$$y = 20 - 66 \sin 2\theta.$$

The curve was plotted down on "papier millimétrique," and has been copied on the same size on the plate. The constants were selected after considerable trial, to give a curve that should be at once characteristic, and of manageable dimensions. The y of the asymptote is 70·6 millimeters.

The distribution of pitch upon the screws of the cylindroid is of fundamental importance in the theory, so that we must study the pitches appropriate to the several points on the cubic.

Let p denote the pitch; then, from the known property of the cylindroid,

$$p = p_0 + m \cos 2\theta,$$

where p_0 is a constant; or, if we express this result by the co-ordinates of the point on the cubic, we have

$$p = p_0 - m \frac{(x^2 - y^2 \cos^2\beta) \cos 2\alpha + 2xy \cos\beta \sin 2\alpha}{x^2 + y^2 \cos^2\beta}.$$

§ 2. **Chord Joining Two Screws of Equal Pitch.**—As the pitches of the two screws, defined by $+\theta$ and $-\theta$, are equal, the chord in question is found by drawing the line through the points x', y' and x'', y'', respectively, where

$$x' = y' \cos\beta \tan(\theta - \alpha),$$

$$y' = h \sec\beta - m \operatorname{cosec}\beta \sin 2\theta,$$

$$x'' = y'' \cos\beta \tan(-\theta - \alpha),$$

$$y'' = h \sec\beta + m \operatorname{cosec}\beta \sin 2\theta.$$

After a few reductions, the required equation is found to be

$$xm(\cos 2\theta + \cos 2\alpha) + y(h \sin\beta + m \cos\beta \sin 2\alpha)$$
$$- h^2 \tan\beta + m^2 \cot\beta \sin^2 2\theta = 0.$$

If this chord passes through the origin, then

$$- h^2 \tan^2\beta + m^2 \sin^2 2\theta = 0;$$

or,

$$h \tan\beta \pm m \sin 2\theta = 0.$$

But this is obviously necessary; for, if either of the equations just written be fulfilled, the ray θ must, of course, intersect the plane at the origin.

We can also determine the chord in a rather longer way, which has, however, the advantage of giving some other expressions that will be of service.

Let $U = 0$ be the cubic curve,

,, $V = 0$ be the equation of the two rays from the origin to the points of intersection of the two equal pitch screws $\pm\theta$.

Let $L = 0$ be the chord joining the two intersections of U and V, distinct from the origin: this is, of course, the chord now sought for. Then we must have an identity of the type

$$cU = VX + LY;$$

for the conditions $L = 0$ and $V = 0$ imply $U = 0$ where c is some constant. L cuts U in three points, two of which lie on V, and the third point, called I, must lie on X. The line X is otherwise arbitrary, and we may, for convenience, take it to be the line ΩI from the origin to I. The product VX thus contains only terms of the third degree, and accordingly the terms of the second degree in U must be sought in LY.

Let $U = u_3 + u_2$ where u_3 and u_2 are of the third and second degrees respectively, then cu_2 must be the quadratic part of the product LY. As L does not pass through the origin, it must have an absolute term, consequently Y must not contain either an absolute term or a term of the first degree. If, therefore, c be the absolute term in L, it is plain that Y must be simply u_2, and we have accordingly,

$$c(u_3 + u_2) = VX + (L' + c)\, u_2,$$

where L' denotes the value of L without the absolute term: we have consequently the identity

$$cu_3 = VX + L'u_2.$$

In this equation we know u_2, u_3, V, and the other quantities have to be found.

If we substitute

$$x = -y \cos\beta \tan(\alpha \pm \theta),$$

we make V vanish, and representing L' by $\lambda x + \mu y$, we find,

$$\lambda \cos\beta \tan(\alpha \pm \theta) - \mu = c \frac{\sin\beta}{h \tan\beta \pm m \sin 2\theta};$$

whence,

$$\frac{\lambda}{c} = \frac{m \cos 2\alpha + m \cos 2\theta}{-h^2 \tan\beta + m^2 \cot\beta \sin^2 2\theta},$$

$$\frac{\mu}{c} = \frac{h \sin\beta + m \cos\beta \sin 2\alpha}{-h^2 \tan\beta + m^2 \cot\beta \sin^2 2\theta}.$$

We can also obtain X; for, let it be $Px + Qy$, then severally identifying the coefficients of x^3 and y^3, we have, after a few reductions,

$$P = m \sin 2\alpha + h \tan\beta,$$

$$Q = -m \cos\beta (\cos 2\alpha + \cos 2\theta);$$

finally, resuming the various results, we have the identity,

$$cU = VX + LY;$$

wherein,

$$c = -h^2 \tan\beta + m^2 \cot\beta \sin^2 2\theta,$$

$$U = \sin\beta \cot^2\beta y^3 + \sin\beta y x^2,$$
$$-(m \sin 2\alpha + h \tan\beta) x^2 + 2mxy \cos\beta \cos 2\alpha$$
$$+ (m \sin 2\alpha \cos^2\beta - h \sin\beta \cos\beta) y^2.$$

$$V = mx^2 (\cos 2\alpha + \cos 2\theta) + 2mxy \sin 2\alpha \cos\beta + my^2 \cos^2\beta (\cos 2\theta - \cos 2\alpha),$$

$$X = x (m \sin 2\alpha + h \tan\beta) - my \cos\beta (\cos 2\alpha + \cos 2\theta),$$

$$L = mx (\cos 2\alpha + \cos 2\theta) + y (h \sin\beta + m \cos\beta \sin 2\alpha)$$
$$- h^2 \tan\beta + m^2 \cot\beta \sin^2 2\theta,$$

$$Y = x^2 (-m \sin 2\alpha + h \tan\beta) + 2mxy \cos\beta \cos 2\alpha$$
$$+ y^2 (m \sin 2\alpha \cos^2\beta - h \sin\beta \cos\beta).$$

§ 3. **Screws Reciprocal to the Cylindroid.**—The screws reciprocal to a cylindroid intersect two screws of equal pitch on the surface. Any chord in the section which cuts the cubic in two points of equal pitch must thus be the residence of a screw reciprocal to the surface; accordingly, the chord

$$mx(\cos 2\alpha + \cos 2\theta) + y\,(h \sin\beta + m \cos\beta \sin 2\alpha) - h^2 \tan\beta + m^2 \cot\beta \sin^2 2\theta = 0,$$

when it receives a pitch equal to

$$-p_0 - m \cos 2\theta,$$

forms a screw reciprocal to the surface.

We have already pointed out (*Theory of Screws*, p. 147) that the envelope of this chord must be a parabola. We can now conveniently obtain the equation to this envelope.

Differentiating with respect to θ, we find

$$x = 2m \cot\beta \cos 2\theta.$$

Eliminating θ, we obtain, as the equation to the parabola,

$$x^2 + 4mx \cot\beta \cos 2\alpha + 4y\,(h \cos\beta + m \cos\beta \cot\beta \sin 2\alpha) - 4h^2 + 4m^2 \cot^2\beta = 0.$$

The vertex of the parabela lies at the point

$$x = -2m \cot\beta \cos 2\alpha,$$

$$y = h \sec\beta - m \operatorname{cosec}\beta \sin 2\alpha.$$

The focal length is

$$\cos\beta\,(h + m \cot\beta \sin 2\alpha).$$

The values of the two equal pitches on the chord are conveniently obtained from the abscissa of the point in which their chord touches the parabola, for we have

$$p = p_0 + m \cos 2\theta;$$

whence,

$$x = 2\,(p - p_0) \cot\beta,$$

or,

$$p = p_0 + \tfrac{1}{2} x \tan\beta.$$

Each point on the parabola will thus correspond to a pair of screws of equal pitch on the cylindroid.

From any point P, on the cubic, two tangents can be drawn to the parabola. Each of these tangents must intersect the cubic in a pair of equal pitch-screws. One tangent will contain the two screws whose pitch is equal to that of P. The other tangent passes through two screws of equal pitch in the two other points, which, with P, make up the three intersections with the cubic.

As the principal screws of the cylindroid are those of maximum and minimum pitch, respectively, it follows that the tangents at these points will also touch the parabola. These common tangents are shown in the Plate.

§ 4. **The Plane at Infinity.**—From the equation of the cylindroid,

$$z(x^2 + y^2) = 2mxy,$$

it follows that the plane at infinity cuts the surface in three straight lines on the planes,

$$z = 0,$$

$$x \pm iy = 0.$$

The line at infinity on the plane of z is of course intersected by all the real generators of the cylindroid, inasmuch as they are parallel to z. The points at infinity on the planes $x \pm iy = 0$ are each the residence of an imaginary screw, also belonging to the surface. The pitches of both these screws are infinite.

We may arrive at the two infinite pitch-screws on the surface in another way. The equations of a screw are

$$y = x \tan\theta,$$

$$z = m \sin 2\theta,$$

while the pitch is

$$p_0 + m \cos 2\theta.$$

If $\tan\theta$ be either $\pm i$, we find both z infinite and the pitch infinite. We thus see that through the infinitely distant point I, on the nodal line of the cylindroid, two screws belonging to the surface can be drawn, just as at any finite point. The peculiarity of the two screws through I is, that their

pitches are equal, *i. e.* both infinite, and this is not the case with any other pair of intersecting screws.

It is now obvious why the envelope of the last section turned out to be a parabola rather than any other conic section. Every plane section will have the line at infinity for a transversal cutting two equal pitch-screws; the envelope of such transversals must thus have the line at infinity for a tangent, *i. e.* must be a parabola.

The existence of this pair of infinite pitch-screws on the cylindroid is a remarkable circumstance, to which, perhaps, sufficient attention has not been paid in the previous study of the subject.

§ 5. **The Parabola.**—In Plate I. will be found the parabola drawn to scale. The equation employed was,

$$y = -31 - \left(5 + \frac{x}{15}\right)^2.$$

When the figure was complete, it was obvious that the parabola touched the cubic, and thus the following theorem was suggested:—

The parabola, which is the envelope of chords joining screws of equal pitch, touches the cubic in three points.

The demonstration is as follows:—To seek the intersections of the parabola with the cubic, we substitute, in the equation of the parabola, the values

$$x = h \tan(\theta - \alpha) - m \cot\beta \sin 2\theta \tan(\theta - \alpha),$$

$$y = h \sec\beta - m \operatorname{cosec}\beta \sin 2\theta.$$

This would, in general, give an equation of the sixth degree for $\tan\theta$. It will, however, be found in this case that the expression reduces to a perfect square. The six points in which the parabola meets the cubic must thus coalesce into three, of which two are imaginary. The values of θ for these three points are given by the equation

$$h \tan(\theta - \alpha) - m \cot\beta (\sin 2\theta \tan(\theta - \alpha) + 2 \cos 2\theta) = 0.$$

§ 6. **Another Demonstration.**—We can prove, geometrically, that the parabola must touch the cubic at any point where the two curves meet.

In general, a cone of screws reciprocal to the cylindroid can be drawn from any external point. If the point O happen to lie on the cylindroid, the cone breaks up into two planes. The nature of these planes is easily seen. One of them, A, must be the plane perpendicular to the generator through O; the other, B, is the plane containing O, and the screw of equal pitch to that of the screw through O. These planes intersect in a ray, L, and it must first be shown that L is a tangent to the cylindroid.

Any ray intersecting one screw on a cylindroid at right angles must cut the surface again in two screws of equal pitch; consequently L can only meet the surface in two distinct points, each of which has the pitch of the generator through O. It follows that L must intersect the surface at two coincident points O, i. e. that it is a tangent to the cylindroid at O.

Let any plane of section be drawn through O. This plane will, in general, intersect A and B in two distinct rays: these are the two screws reciprocal to the cylindroid, and they are accordingly the two tangents from O to the parabola we have been discussing. The only case in which these two rays could coalesce would occur when the plane of section was drawn through L; but the two tangents to a parabola from a point only coalesce when that point lies on the parabola. At a point where the parabola meets the cubic, L must needs be a tangent both to the parabola and to the cubic, which can only be the case if the two curves are touching. We have thus proved that the parabola must have triple contact with the cubic.

There are thus three points on the cubic which have the property that the tangent intersects the curve again in a point of equal pitch to that of the point of contact. We thus learn that all the screws of a 4-system which lie in a plane touch a parabola having triple contact with the reciprocal cylindroid.

§ 7. **Chord Joining Two Points.**—If θ and θ' be the angles by which two points on the cubic are defined, then the equation to the chord joining those points is

$$Ax + By + C = 0;$$

where
$$A = 2m \cos(\theta' - \alpha) \cos(\theta'' - \alpha) \cos(\theta' + \theta''),$$
$$B = h \sin\beta + m \cos\beta \cos(\theta' + \theta'') \sin(2\alpha - \theta' - \theta'')$$
$$- m \cos\beta \sin(\theta' + \theta'') \cos(\theta' - \theta''),$$
$$C = - \tan\beta\,(h - m \cot\beta \sin 2\theta')\,(h - m \cot\beta \sin 2\theta'').$$

If in these expressions we make $\theta' + \theta'' = 0$, we obtain the equation for the chord joining screws of equal pitch, as already obtained.

We shall find that, in particular sections, these expressions become considerably simplified. Suppose, for example, that the plane of section be a tangent plane to the cylindroid. The cubic then degenerates to a straight line and a conic. The condition for this will be obvious from the equation of the cubic (§ 1). If the coefficient of x^2 become zero, the required decomposition takes place, for y is then a factor. The necessary and sufficient condition for the plane of section being a tangent, is therefore

$$m \sin 2\alpha + h \tan\beta = 0.$$

When this is the case, the three expressions of A, B, C may be relieved from a common factor,

$$2m \cos(\theta' - \alpha) \cos(\theta'' - \alpha),$$

and we have

$$A = \cos(\theta' + \theta''),$$
$$B = - \cos\beta \sin(\theta' + \theta''),$$
$$C = - 2m \cot\beta \sin(\alpha + \theta') \sin(\alpha + \theta'').$$

If the screws be of equal pitch, $\theta' + \theta'' = 0$, the coefficient of y disappears, and we see that all the chords are merely lines parallel to y, which is parallel to one of the axes of the ellipse.

The equation to the chord then becomes

$$(\cos 2\alpha + \cos 2\theta)\,(x + m \cot\beta\,(\cos 2\alpha - \cos 2\theta)) = 0.$$

For a given value of x there are two values of θ corresponding to the two chords that can be drawn through the point. One of these is parallel to y, and has a θ obtained from the equation

$$x + m \cot\beta\,(\cos 2\alpha - \cos 2\theta) = 0.$$

The other value of θ is $\frac{\pi}{2} - \alpha$, from the equation

$$\cos 2\alpha + \cos 2\theta = 0.$$

This is independent of x, as might have been foreseen from the fact that the two screws of equal pitch are in this case the line in the section and its other equal pitch screw. The latter cuts the section in a certain point, and, of course, all chords through this point meet the curve in two screws of equal pitch.

§ 8. **Reciprocal Screws.**—Another important branch of the subject must now be considered. We shall first investigate the following general problem :—

From any point, P, a series of transversals are drawn across each pair of reciprocal screws on the cylindroid. It is required to determine the cone which is the locus of these transversals. We shall show that this is a cone of the second degree.

Let α, β, γ be the co-ordinates of the point. Then the plane through this point, and the generator of the cylindroid defined by the equations

$$y = x \tan \theta,$$

$$z = m \sin 2\theta,$$

is, $$(y - \tan\theta)(\gamma - m \sin 2\theta) = (\beta - \alpha \tan\theta)(z - m \sin 2\theta);$$

or, if we arrange in powers of $\tan\theta$, we obtain

$$A \tan^3\theta + B \tan^2\theta + C \tan\theta + D = 0,$$

in which

$$A = \alpha z - \gamma x; \quad D = \gamma y - \beta z,$$

$$B = \gamma y - \beta z + 2mx - 2m\alpha; \quad C = \alpha z - \gamma x + 2m\beta - 2my.$$

If the same transversal also crosses the generator defined by θ, then,

$$A \tan^3\theta' + B \tan^2\theta' + C \tan\theta + D = 0.$$

When the two screws defined by θ and θ' are reciprocal,

$$\tan\theta \tan\theta' = H,$$

when H is a constant.

By eliminating and rejecting the factor,

$$\tan\theta - \tan\theta',$$

we obtain,

$$-A^2H^3 + ACH^2 - BDH + D^2 = 0.$$

And as this is of the second degree in x, y, z, the required theorem has been proved.

All these cones must pass through the centre of the cylindroid, inasmuch as the two principal screws of the cylindroid are reciprocal. If a constant be added to all the pitches of the screws on the cylindroid, then the pairs of reciprocals alter, inasmuch as H alters. The cone changes accordingly, and thus there would be through each point a family of cones, all of which, however, agree in having, as a generator, the ray from the vertex to the centre of the cylindroid. Thus, even when the cylindroid is given, we must further have the pitch of a stated screw given before the cone becomes definite. This state of things may be contrasted with that presented by the cone of reciprocal screws which may be drawn through a point. The latter depends only upon the cylindroid itself, and is not altered if all the pitches be modified by a constant increment.

The discriminant of the cubic,

$$-A^2H^3 + ACH^2 - BDH + D^2 = 0,$$

is

$$A^2D^2(A^2D^2 + \tfrac{4}{27}B^3D + \tfrac{4}{27}AC^3 - \tfrac{1}{27}B^2C^2 - \tfrac{2}{3}ABCD).$$

Omitting the factor A^2D^2, we have, for the envelope of the system of cones, the cone of the fourth order, found by placing the expression in the bracket equal to zero. It may be noted that the same cone is the envelope of the planes,

$$AH^3 \pm BH^2 + CH \pm D = 0.$$

§ 9. **Application to the Plane Section.**—We are now to study the chord joining a pair of reciprocal points on the cubic. Take any point in the plane of the section; then, as we have just seen, a cone of screws can be drawn through this point, each ray of which crosses two reciprocal screws. This cone is cut by the plane of section in two rays, and, accordingly, we

see that through any point in the plane of section two chords can be drawn through a pair of reciprocal points. The actual situation of these chords is found by drawing a pair of tangents to a certain hyperbola. This will now be proved.

The values of θ and θ', which correspond to a pair of reciprocal points, fulfil the condition,

$$\tan\theta \tan\theta' = H;$$

whence,

$$\cos(\theta - \theta') = \lambda \cos(\theta + \theta');$$

where, for brevity, we write λ instead of

$$\frac{1 + H}{1 - H}.$$

If, further, we make $\theta + \theta' = \psi$, we shall find, for the equation of the chord,

$$Ax + By + Cz = 0;$$

in which,

$$A = \frac{m}{2}(\lambda + \cos 2\alpha) + \frac{m}{2}\sin 2\alpha \sin 2\psi + \frac{m}{2}(\lambda + \cos 2\alpha)\cos 2\psi,$$

$$B = h \sin\beta + \frac{m}{2}\cos\beta \sin 2\alpha - \frac{m\cos\beta}{2}(\lambda + \cos 2\alpha)\sin 2\psi$$

$$+ \frac{m}{2}\cos\beta \sin 2\alpha \cos 2\psi,$$

$$C = -h^2 \tan\beta - \frac{m^2}{2}(\lambda^2 - 1)\cot\beta + \lambda hm \sin 2\psi - \frac{m^2}{2}(\lambda^2 - 1)\cot\beta \cos 2\psi.$$

The envelope of this chord is found to be

$$\begin{aligned}
0 = &+ x^2 \,|\, \tfrac{1}{4}m^2 \sin^2 2\alpha, \\
&+ y^2 \,|\, \tfrac{1}{4}m^2 \cos^2\beta \cos^2 2\alpha - hm \sin\beta \cos\beta \sin 2\alpha - h^2 \sin^2\beta \\
&\qquad + \tfrac{1}{2}m^2 \lambda \cos 2\alpha \cos^2\beta + \tfrac{1}{4}m^2 \lambda^2 \cos^2\beta, \\
&+ xy \,|\, -\frac{m^2}{2}\sin 2\alpha \cos 2\alpha \cos\beta - mh \cos 2\alpha \sin\beta \\
&\qquad - \tfrac{1}{2}m^2 \lambda \sin 2\alpha \cos\beta - mh\lambda \sin\beta, \\
&+ x \,|\, + mh^2 \cos 2\alpha \tan\beta + hm^2 \lambda \sin 2\alpha + mh^2 \lambda \tan\beta, \\
&+ y \,|\, - m^2 h \cos\beta + mh^2 \sin\beta \sin 2\alpha + 2h^3 \sin\beta \tan\beta - m^2 h\lambda \cos\beta \cos 2\alpha, \\
&+ \,|\, m^2 h^2 - h^4 \tan^2\beta.
\end{aligned}$$

Using the data already assumed for Plate I., and, with the addition now made of taking λ to be $-\frac{1}{3}$, the equation reduces to

$$122x^2 - 203y^2 - 161xy - 2417x - 5003y + 245436 = 0;$$

which, for convenience of calculation, I change into

$$x = 1{\cdot}6 + 21{\cdot}6 \sec\phi \pm 47{\cdot}5 \tan\phi,$$

$$y = -12{\cdot}8 + 32{\cdot}8 \sec\phi.$$

This hyperbola has been plotted down in Plate I. It obviously touches the cubic at three points. I had not anticipated this until the curves were carefully drawn; but, when the theorem was suggested in this manner, it was easy to provide the following demonstration :—

§ 10. **Proof that the Hyberbola which is the Envelope of the Chords joining Reciprocal Points has Triple Contact with the Cubic.**—The cone of reciprocal chords drawn through any point P breaks up into a pair of planes when P lies on the cylindroid. (I use the expression, reciprocal *chords*, to signify the transversal drawn across a pair of reciprocal screws on the cylindroid. This is very different from a screw reciprocal to the cylindroid.) For, take the screw reciprocal to that which passes through P. Then the plane X, through P and this screw, is obviously one factor of the locus. Draw through P any transversal across a pair of reciprocals on the cylindroid, then the plane Y, through the centre and this transversal, will be the other factor of the locus. This pair of planes, X and Y, intersect in a ray which we shall call M.

A plane of section through P will, of course, usually cut the two planes in two rays, and these will be the two reciprocal chords through P. But suppose the plane of section happened to pass through M, then there will be only one reciprocal chord through P, and this will, of course, be M. Now, M must be a tangent to the cylindroid at P. Every chord through P, in the plane of Y, must cut the surface again in a pair of reciprocal points. To this M must be no exception, and as it lies in X, it intersects the screw reciprocal to that through P; therefore the third intersection of M with the

surface must coalesce with P, or, in other words, M must be a tangent to the surface at P. We have thus shown that in the case where two reciprocal chords through a point on the cubic coalesce into one, that one must be the tangent to the cubic at P.

But the two reciprocal chords through a point will only coalesce when the point lies on the hyperbola, in which case the two chords unite into the tangent to the hyperbola. Consider, then, the case where the hyperbola meets the cubic at a point P, inasmuch as P lies on the hyperbola, the two chords coalesce into a tangent thereto, but because they do coalesce, this line must needs be also a tangent to the cubic; hence, whenever the hyperbola meets the cubic the two curves must have a common tangent. Altogether the curves meet in six points, which unite into three pairs, thus giving the required triple contact between the hyperbola and the cubic.

It will be noticed that the reasoning is analogous to that used in demonstrating that the parabola had triple contact in § 6.

§ 11. **Family of Hyperbolæ.**—The variations of λ produce no alteration in the cubic; but, of course, the hyperbola varies with each change of λ. In every case, however, it has the triple contact, and there is also a fixed tangent which must touch every hyperbola. This is the chord joining the two principal screws on the cylindroid; for, as these are reciprocal, notwithstanding any augmentation to the pitches, their chord must always touch the hyperbola. The family of Hyperbolæ, corresponding to the variations of λ, is thus concisely represented, as they must all touch this fixed line, and have triple contact with a fixed curve: that is, they must each fulfil four conditions, leaving one more disposable quantity to make up the five necessary for the complete definition of a conic.

§ 12. **Discriminant of the Hyperbola.**—We write the tangent to the hyperbola or the reciprocal chord in the form

$$L \cos 2\psi + M \sin 2\psi + N = 0.$$

If a pair of values can be found for x and y, which will simultaneously satisfy

$$L = 0, \quad M = 0, \quad N = 0,$$

then every chord of the type

$$L \cos 2\psi + M \sin 2\psi + N = 0$$

must pass through this point. The condition for this is, that the discriminant of the hyperbola is zero, and we find the discriminant to be

$$\tfrac{1}{4} m^2 h \sin\beta (\lambda^2 - 1)(h \tan\beta + m \sin 2\alpha).$$

There are two critical cases in which this expression vanishes. It does so if

$$h \tan\beta + m \sin 2\alpha = 0;$$

where, as we have already seen, the plane of section is tangential to the cylindroid—an important case, which we shall discuss subsequently.

But we also note that the discriminant will vanish if

$$h = 0,$$

i. e. if the plane of section passes through the centre of the cylindroid.

We might have foreseen this from the results of the last article; for the plane of Y is a central section, and the hyperbola has evidently degenerated, for all the reciprocal chords, instead of touching an hyperbola, merely pass through the common apex P. The case of the central section is one of considerable interest, and will repay examination.

§ 13. **The Central Section of the Cylindroid.**—By this we mean a section of the surface, special in no other sense, save that it passes through the centre of the surface. The equation to the central section is

$$y^3 \sin\beta \cos^2\beta + yx^2 \sin\beta - mx^2 \sin 2\alpha + 2mxy \cos\beta \cos 2\alpha + my^2 \sin 2\alpha \cos^2\beta = 0.$$

The chord joining points of equal pitch $+\theta$ is

$$x(\cos 2\theta + \cos 2\alpha) + y \cos\beta \sin 2\alpha + m \cot\beta \sin^2 2\theta = 0.$$

The apex P through which all reciprocal chords pass is

$$x = \frac{m(\lambda^2 - 1)\cot\beta\,(\lambda + \cos 2\alpha)}{1 + 2\lambda \cos 2\alpha + \lambda^2},$$

$$y = \frac{m(\lambda^2 - 1)\operatorname{cosec}\beta \,.\, \sin 2\alpha}{1 + 2\lambda \cos 2\alpha + \lambda^2};$$

and in general the co-ordinates of a point on the cubic are

$$x = y \cos\beta \tan(\theta - \alpha),$$
$$y = -m \operatorname{cosec}\beta \sin 2\theta.$$

In Plate II. we have shown one of these curves drawn to scale. It was obtained from the equations

$$x = \cdot 9y \tan(\theta - 25),$$
$$y = -66 \sin 2\theta.$$

The parabola is also shown, which is the envelope of equal pitch-chords, the equation being

$$y = -51 - \left(6 + \frac{x}{13}\right)^2.$$

The principal screws on the cylindroid both pass through the double point O; the two tangents to the curve at this point must therefore both touch the parabola.

The leading feature of the central section is expressed by the important property possessed by the chords joining reciprocal screws. If we add any constant to the pitches, then we alter λ, and, accordingly, the point P, through which all reciprocal chords pass, moves along the curve.

The tangent to the cubic at P meets the cubic again in the point reciprocal to P. Two tangents, real or imaginary, can be drawn from P to the cubic touching it in the points T_1, T_2, respectively: as these must each correspond to a screw reciprocal to itself, it follows that T_1 and T_2 are the screws of zero pitch. We hence see that the two tangents from any point on the cubic touch the cubic in points of equal pitch.

Let α and β be two screws, and γ and δ another pair of screws, and let the two chords, $\alpha\beta$ and $\gamma\delta$, intersect again on the cubic. If d and θ be the perpendicular distance and angle between the first pair, and d' and θ' the corresponding quantities for the second pair, then there must be some quantity ω, which, if added to all the pitches on the cylindroid, will make α and β reciprocal, and also γ and δ reciprocal. We thus have

$$(p_\alpha + p_\beta + 2\omega)\cos\theta - d\sin\theta = 0,$$
$$(p_\gamma + p_\delta + 2\omega)\cos\theta' - d'\sin\theta' = 0;$$

whence, $$p_\alpha + p_\beta - d \tan\theta = p_\gamma + p_\delta - d' \tan\theta';$$

in other words, for every pair of screws, α and β, whose chords belong to a pencil diverging from a common point on the surface, the expression

$$p_\alpha + p_\beta - d \tan\theta$$

is a constant. The value of this constant is double the pitch of the screw of either of the points of contact of the two tangents from P to the curve.

§ 14. **Relation between Two Conjugate Screws of Inertia.**—We have found the relation between a pair of conjugate screws of inertia so important in the dynamical part of the theory, that it is worth while to investigate the properties of the chord joining two such points in the central section. It will first be necessary to find the relation between a pair of angles, ϕ and θ, corresponding to two conjugate screws of inertia.

This is most readily obtained from the circular representation, for the chord joining the conjugate points of inertia passes always through a fixed point.

Let A and B be the principal screws, and let XY be a chord joining

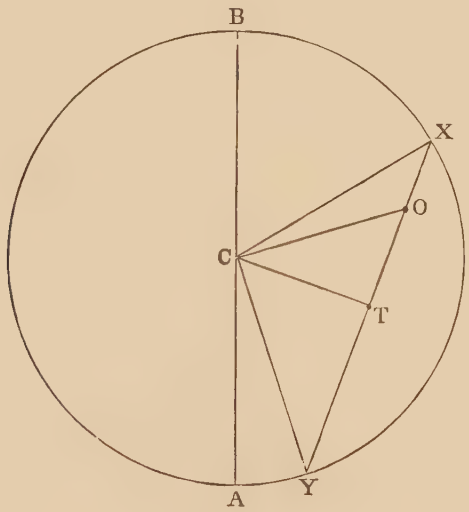


Fig. 3.

conjugate points of inertia X and Y, which constantly pass through the fixed point O (fig. 3). Then if CT be the perpendicular on XY, and the angle

$XCA = 2\phi$, the angle $YCA = 2\theta$, and the angle $OCA = \omega$, we have, from the figure,

$$CX \cos(\phi - \theta) = CO \cos(\phi + \theta - \omega).$$

Substituting this expression for $\cos(\phi - \theta)$ in the equation of the chord in § 6, we see that the chord must always envelope a conic. This conic, and the point P on the cubic through which all reciprocal chords will pass, will enable the impulsive screw, corresponding to any instantaneous screw, to be immediately determined. For, draw through any point S that tangent to the conic which gives S as one of the two conjugate screws of inertia which must lie upon it; let S' be the othe rconjugate screw; then the chord PS' will cut the cubic again in the required impulsive screw. The two principal screws of inertia are found by drawing from P that tangent to the conic which has *not* P as one of the two conjugate screws of inertia The two intersections of this tangent, with the cubic, are the required principal screws of inertia.

§ 15. **Application to the General Case.**—It is also possible to determine, with regard to any section, the relation between the impulsive screw and the instantaneous screw. We have here to consider two conics connected with the cubic, viz. the reciprocal conic, which is the envelope of reciprocal chords, and the inertia conic, which is the envelope of chords of conjugate screws of inertia. We must provide a means of discriminating the two tangents from a point P on the cubic to either conic; any ray, of course, cuts the cubic in three points, of which two possess the characteristic relation. If P be one of these two, we may call this tangent the '*odd* tangent.' The other tangent will have, as its significant points, the two remaining intersections; leaving out P, we can then proceed, as follows, to determine the impulsive screw corresponding to P as the instantaneous screw :—

Draw the odd tangent from P to the inertia conic, and from the conjugate point thus found draw the odd tangent to the reciprocal conic. The reciprocal point Q thus found is the impulsive screw corresponding to P as the instantaneous screw.

In general there are four common tangents to the two conics. Of these

tangents there is only one possessing the property, that the same two of its three intersections with the cubic are the correlative points with respect to each of the conics. These two intersections are the principal screws of inertia.

§ 16. **Small Oscillations.**—To determine the small oscillations we find the *potential* conic, the tangents to which are chords joining two conjugate screws of the potential (*Theory of Screws*, p. 66). The two harmonic screws are then to be found on one of the two common tangents to the two conics. It can be shown that both the inertia conic and the potential conic will, like the reciprocal conic, have triple contact with the cubic.

§ 17. **Section Parallel to the Nodal Line.**—If the node on the cubic be at infinity, the form of equation to the cubic hitherto employed will be illusory. The nature of this section must therefore be studied in another way, as follows:—

Let the plane cut the two perpendicular screws in A and B. Let l be the perpendicular OC from O upon C, and let η be the inclination of this

Fig. 4.

perpendicular to the axis of x. Then, taking OA as the new axis of x, in which case z will be the new y, we have,

$$x' = l \tan(\eta - \theta),$$
$$y' = m \sin 2\theta.$$

Eliminating θ, and omitting the accents, we have, as the equation of the cubic,

$$yx^2 + mx^2 \sin 2\eta + l^2 y + 2lmx \cos 2\eta - ml^2 \sin 2\eta = 0.$$

The chord joining the two points θ and θ' has, as its equation,

$$mx \cos(\theta + \theta')[\cos(2\eta - \theta - \theta') + \cos(\theta - \theta')] + ly,$$

$$- ml \sin(2\eta - \theta - \theta') \cos(\theta + \theta') - ml \cos(\theta - \theta') \sin(\theta + \theta') = 0.$$

If this be the chord joining the screws of equal pitch, then

$$\theta + \theta' = 0,$$

and the equation reduces to

$$mx(\cos 2\eta + \cos 2\theta) + ly - ml \sin 2\eta = 0.$$

We thus see that this chord, which in the general section envelops a parabola, now passes constantly through the fixed point

$$x = 0,$$

$$y = + m \sin 2\eta.$$

This result could have been foreseen; for, consider that screw on the cylindroid (and there must always be one) normal to the plane which it intersects at a point P, any ray in the plane through P is perpendicular to this screw, and therefore, by a well-known property of the cylindroid, must intersect the curve again in two points of equal pitch. This point P is, of course, the point whose existence we have demonstrated above.

§ 18. **On the Tangential Section of the Cylindroid.**—In this case the cubic degenerates to the product of a straight line and an ellipse. The case is one of special interest and importance, for the ellipse really leads to that circular representation of the cylindroid on which I have already communicated to this Academy.

We shall first investigate, in a somewhat novel manner, the distribution of pitch upon the several screws corresponding to the points on the ellipse. We shall be guided by the following principle:—Let α and β be the two screws of zero pitch, and let θ be any screw of zero pitch intersecting both α and β, and not, of course, lying on the cylindroid. The screw θ is reciprocal both to α and to β, and is therefore reciprocal to every screw on the

cylindroid determined by α and by β. We therefore have, for any screw ρ of the surface, the following condition :—

$$p_\rho \cos \omega - d \sin \omega = 0 ;$$

where ω is the angle between ρ and θ, and d is their shortest distance. We thus find the pitch by the equation

$$p_\rho = d \tan \omega.$$

To represent the pitch by a geometrical construction we must proceed as follows :—

Let OH (fig. 5) be the projection of the nodal line of the cylindroid on the plane of section, and suppose ϵ to be the inclination of the axis to OH.

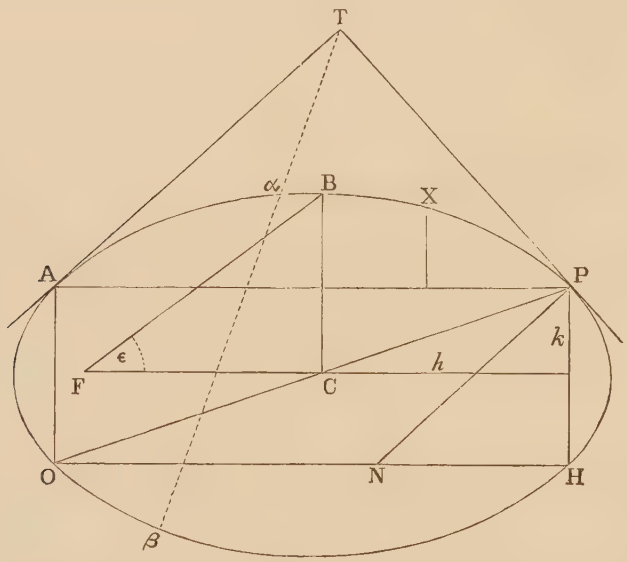


Fig. 5.

Let OA, in the plane of section, be a generator of zero pitch on the surface. Then, since OA is perpendicular to the axis it will be perpendicular to OH. The section of the cylindroid by the plane consists of the ray OA and the ellipse $OAPH$. Let P be the point in which the second screw of zero pitch cuts the ellipse. A ray through P, in the plane of section, must be always perpendicular to the screw at the other point where it intersects

the ellipse. Join PH, then a perpendicular from H on the nodal axis will be a screw of the system, and is therefore perpendicular to PH; hence the angle at H is a right angle. Any chord drawn through two screws of equal pitch must be perpendicular to the third screw it intersects, that is, to AO. But P and A are equal pitch-points, and, accordingly, AP must be perpendicular to AO. The figure $APHO$ is therefore a rectangle.

The angle ϵ is connected with the dimensions of the ellipse. A tangent at P must be perpendicular to the perpendicular let fall from P on the axis. Draw the normal PN at P, and erect at N a perpendicular to the plane intersecting the axis at N' (not, of course, shown in the figure). Then $N'P$ is perpendicular to the tangent at P and to ON'.

Whence we have,

$$OP \cos PON' = ON';$$

but,

$$\cos PON' = \cos PON \cos \epsilon,$$

and,

$$ON' \cos \epsilon = ON, \quad \text{and } ON \cos PON = 2h,$$

where h and k are the co-ordinates of P; whence we obtain

$$h \cos 2\epsilon = CN,$$

$$= h\left(1 - 2\frac{b^2}{a^2}\right);$$

and, finally,

$$\sin^2 \epsilon = \frac{b^2}{a^2}.$$

Thus we see that ϵ depends only on the axis of the ellipse. If the focus F be joined to the extremity of the minor axis B, the angle BFC is equal to ϵ.

Let X be any point of the ellipse, and it is required to find the pitch of the screw passing through X.

A ray through X, perpendicular both to XP and to the axis, will be a screw on the cylindroid, and the moment of this line with regard to OP,

divided by the cosine of the angle between the two, is the pitch. The direction cosines of the screw are

$$R \sin \epsilon (y' - k), \quad - R \sin \epsilon (x' - h), \quad - R \cos \epsilon (y' - k),$$

and it passes through the point x', y', 0. The direction cosines of OP are,

$$R' h, \quad R' k, \quad O,$$

where R and R' are factors introduced to make the sum of the squares unity.

After a few reductions, the pitch assumes the elegant shape

$$(k - y') \cot \epsilon.$$

In other words, the pitch only differs by the constant factor $\cot \epsilon$ from the perpendicular distance of the point from AP.

This exhibits the distribution of pitch in a very clear manner. An ellipse is drawn, and a ray parallel to the major axis. Then the pitch of each screw is equal to the perpendicular distance of the corresponding ray from each point multiplied into $\cot \epsilon$. The various screws on the cylindroid may be concisely represented by their corresponding points on the ellipse.

Each screw on the cylindroid will have one other screw on the surface reciprocal thereto. To each point on the ellipse there will, therefore, be a corresponding point, so that the two shall indicate a pair of reciprocal screws. The geometrical relation of these pairs is of interest, and we have now to prove the following theorem :—

Any ray passing through the pole of the pitch-line intersects the ellipse in a pair of points corresponding to two reciprocal screws.

Let R and R' denote the reciprocals of the distances of points $x'y'$ and $x''y''$ from P. The moments of the two screws will be

$$RR' \sin \epsilon \cos \epsilon (x' - x'')(- x'y'' + x''y' + hy'' - hy' + kx' - kx'');$$

but this is to be equal to the sums of the pitches into the cosine of the angle, that is, to

$$RR' \cot \epsilon (2k - y' - y'')[(y' - k)(y'' - k) + \sin^2 \epsilon (x' - h)(x'' - h)].$$

If we reduce their expressions by introducing the eccentric angles α, β, θ for x', y'; x'', y''; P, respectively, we obtain the condition

$$\sin\theta \cos\tfrac{1}{2}(\alpha - \beta) - \sin\tfrac{1}{2}(\alpha + \beta) = 0.$$

But this is easily seen to be equivalent to the statement that the points α and β are collinear with the point T in which the tangent at P intersects the minor axis. The theorem is therefore proved. It may be stated more concisely, as follows:—A pair of harmonic conjugates to the extremities of the pitch-line correspond to a pair of reciprocal screws.

The correspondence between the several points upon the ellipse, and the several screws on the cylindroid, can be exhibited in a very clear manner.

We know that the pitch of any screw is represented as follows:—

$$p - m \cos 2\omega;$$

where p is an arbitrary constant, $2m$ is the length of the axis of the cylindroid, and ω the angle between the screw and the axis of reference. ("Theory," chap. II.).

Identifying with the expression found in this chapter, we have

$$p - m \cot 2\omega = \cot\epsilon\,(k - y').$$

As these expressions must be always true, we shall have, for the maximum and minimum pitches,

$$p - m = \cot\epsilon\,(k - b),$$

$$p + m = \cot\epsilon\,(k + b);$$

whence, in general,

$$p = k \cot\epsilon; \quad m = b \cot\epsilon;$$

whence,

$$\cos 2\omega = \frac{y}{z}.$$

This gives the following construction for the angle between two screws, corresponding to points P and Q. Draw the circle (fig. 6) which has the major axis of the ellipse for its diameter, and take on this the points P' and Q', found by letting fall perpendiculars from P and Q on the diameter. Then

the angle between the screws is the angle subtended in the circle by the arc $P'Q'$

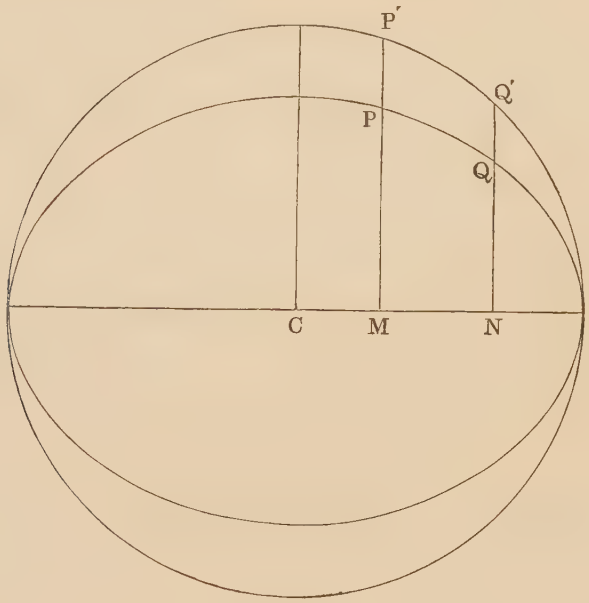


Fig. 6.

The perpendicular distance between the two screws is also exhibited in a very clear manner. It is equal to the difference of the expressions in $\sin 2\omega$ for the two screws "Theory," chap. II.). This is easily seen to be equal to the very simple expression

$$e \,.\, MN,$$

where e is the eccentricity of the ellipse.

We may test these results by calculating the virtual coefficient of the two screws. The required expression is the half of

$$(p_\alpha + p_\beta) \cos\phi - d \sin\phi,$$

where ϕ is the angle between the screws, and d their perpendicular distance. As before, we make $k = b \sin\theta$. Then we have the following :—

$$p_\alpha = ae(\sin\theta - \sin\alpha); \quad p_\beta = ae(\sin\theta - \sin\beta),$$

$$\cos\phi = \cos\tfrac{1}{2}(\alpha - \beta), \quad d = ae(\cos\beta - \cos\alpha);$$

whence we deduce for the virtual coefficient

$$m\left(\sin\theta\cos\tfrac{1}{2}(\alpha-\beta)-\sin\tfrac{1}{2}(\alpha+\beta)\right).$$

If $\alpha=\beta$, the expression reduces to the pitch, as it obviously ought.

The evanescence of the virtual coefficient gives the condition, that the two screws, α and β, shall be reciprocal. This condition has been already found. It may be written,

$$\sin\theta=\frac{\tan\frac{1}{2}\alpha+\tan\frac{1}{2}\beta}{1+\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}.$$

The angle between the two screws, α and β, can also be exhibited in a projective manner. Form the anharmonic ratio of the two points, α and β, and the two points at infinity lying on the ellipse, then the angle between the two screws is equal to the logarithm of this ratio divided by $2\sqrt{-1}$.

§ 19. **The Circular Representation.**—We have seen how the several

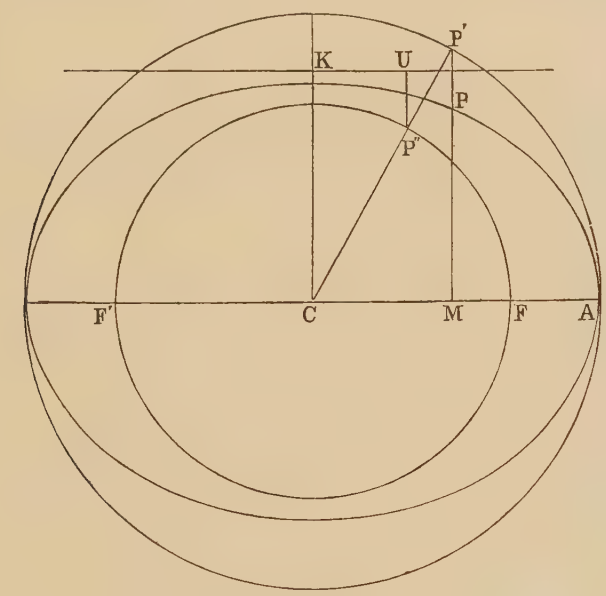


Fig. 7.

screws on the cylindroid correspond to the points on an ellipse. These points, in turn, correspond to other points on the circumference of the

circle described on the major axis as diameter. There is yet one more stage to be reached in the process of simplifying the correspondence. For this we use the circle of which the line joining the foci is the diameter (fig. 7).

Let P be a point on the ellipse, P' the corresponding point on the large circle, and P'' the point on the circle described on the line FF' joining the foci, as diameter. Then, as P travels round the ellipse, we may regard the point P'' as corresponding to the screw which passes through P.

The pitch of this screw is

$$\cot \epsilon (k - y);$$

or, taking p as the constant part of the pitch,

$$p - y \cot \epsilon;$$

and, after a few reductions, and bearing in mind that $\cot \epsilon = ea \div b$, we have for the pitch the simple expression

$$p - q,$$

where q is the perpendicular distance from P'' on the axis AB. If, then, the line UK be drawn parallel to the major axis, and at the distance p therefrom, we see that the pitch of the screw is simply equal to the perpendicular distance from P'' thereto.

The angle between two screws through P and through Q is equal to the angle subtended in the circle by the arc $P''\,Q''$, while the perpendicular distance of the two screws is equal to the distance between the feet of the perpendiculars from P'' and Q'' on the major axis.

Much simplicity has thus been gained by the choice of this circle. Notice also that whatever be the tangent plane cutting the cylindroid, and however variable be the ellipse produced by the section, the distance between the foci, and therefore the dimensions of this circle, is constant. The pitch-line KU is also at a constant distance from the centre of the circle, whatever be the plane in which the section is made.

We find in this circle, and the corresponding pitch-line, great facilities for the study of the dynamical problems relating to freedom of the second

order. But it must not be supposed that the representation of the cylindroid in this manner requires all the discussion we have here given. The method we have followed exhibits the origin and geometrical relations of the correspondence; but it can be established in a much more direct manner, as has already been shown to this Academy. (See "Cunningham Memoirs," No. IV.)

EXPLANATION OF PLATE II.

General Section of the Cylindroid, showing—

(1) Cubic with the double point O.

(2) Asymptote of the cubic.

(3) The parabola, which is the envelope of the chords joining screws of equal pitch.

(4) Hyperbola having triple contact with the cubic, being envelope of reciprocal chords.

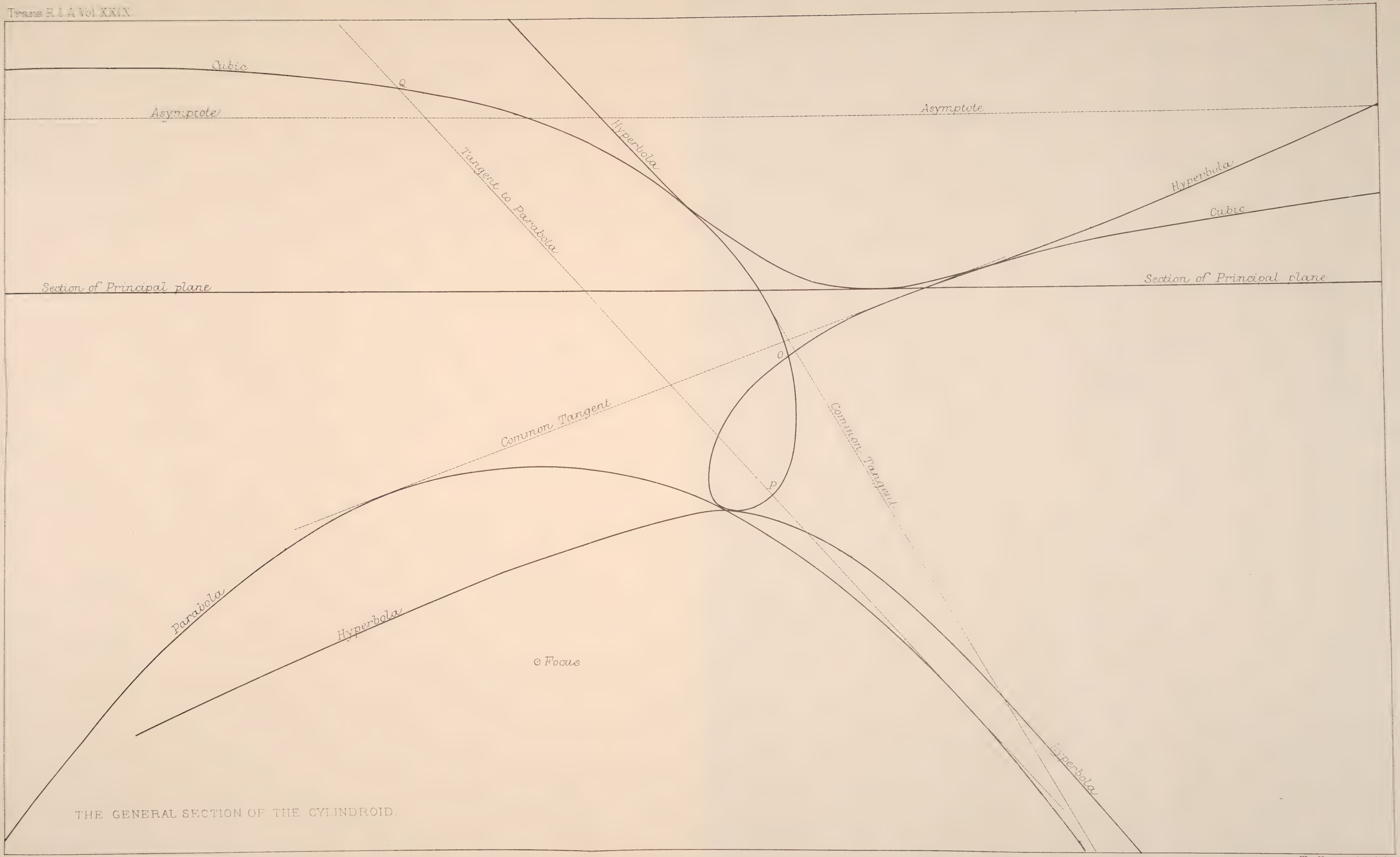
(5) Section of principal plane. Also tangent to the hyperbola.

(6) A tangent to the parabola, showing two screws, P and Q, of equal pitch.

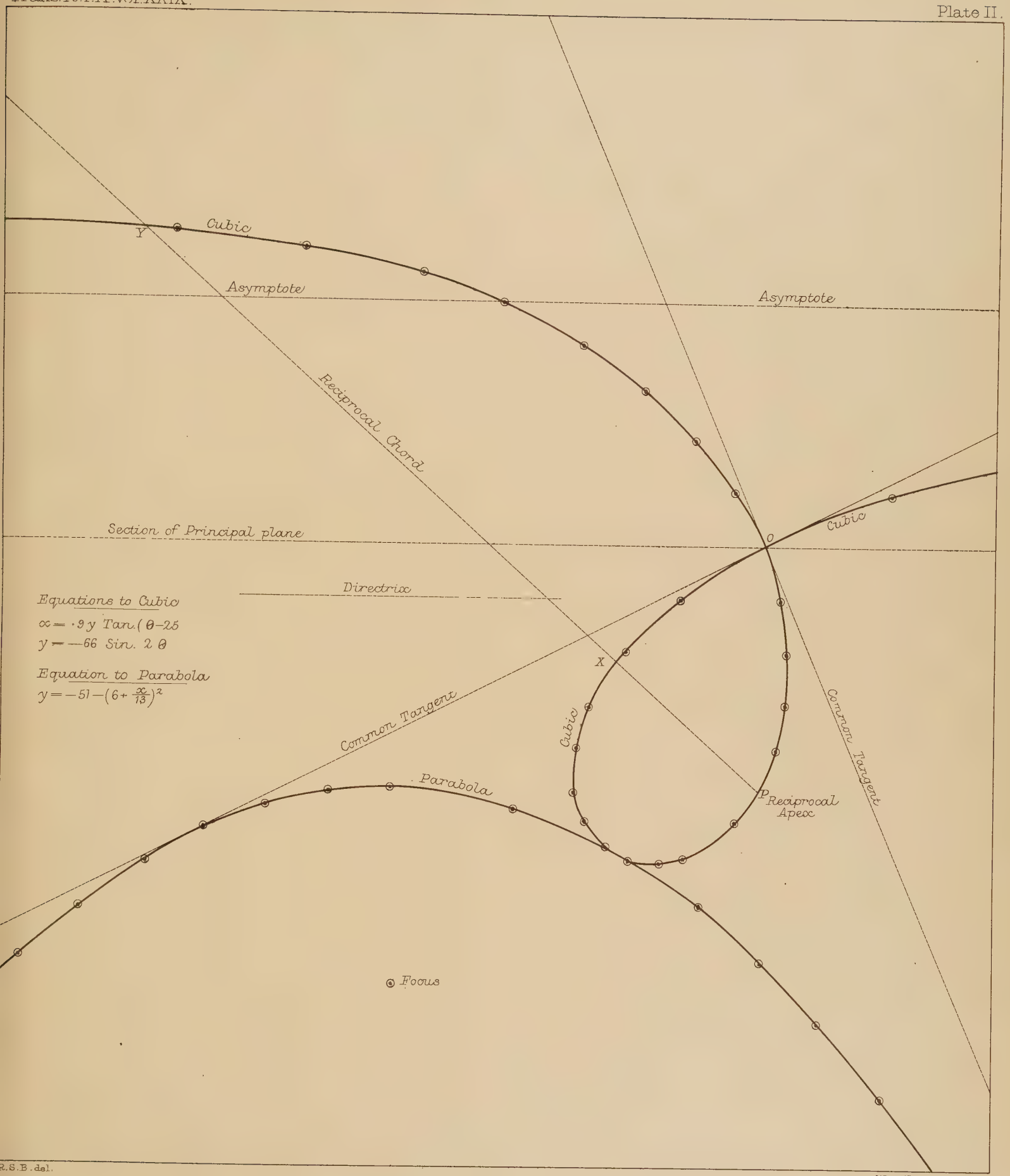
(7) Common tangents to the parabola and the cubic, touching the latter at the two principal screws.

(8) Any tangent to the hyperbola intersects the cubic in three points, two of which belong to reciprocal screws (not shown).

Equations of Cubic.	*Equation of Parabola.*	*Equations of Hyperbola.*
$x = \cdot 9y \tan(\theta - 25)$, $y = 20 - 66 \sin 2\theta$.	$y = -31 - \left(5 + \frac{x}{15}\right)^2$.	$x = 1\cdot 6 \pm 21\cdot 6 \sec\phi \pm 47\cdot 5 \tan\phi$, $y = -12\cdot 8 + 32\cdot 8 \sec\phi$.



THE GENERAL SECTION OF THE CYLINDROID.



R.S.B. del.

West, Newman & Co. lith.

THE CENTRAL SECTION OF THE CYLINDROID.

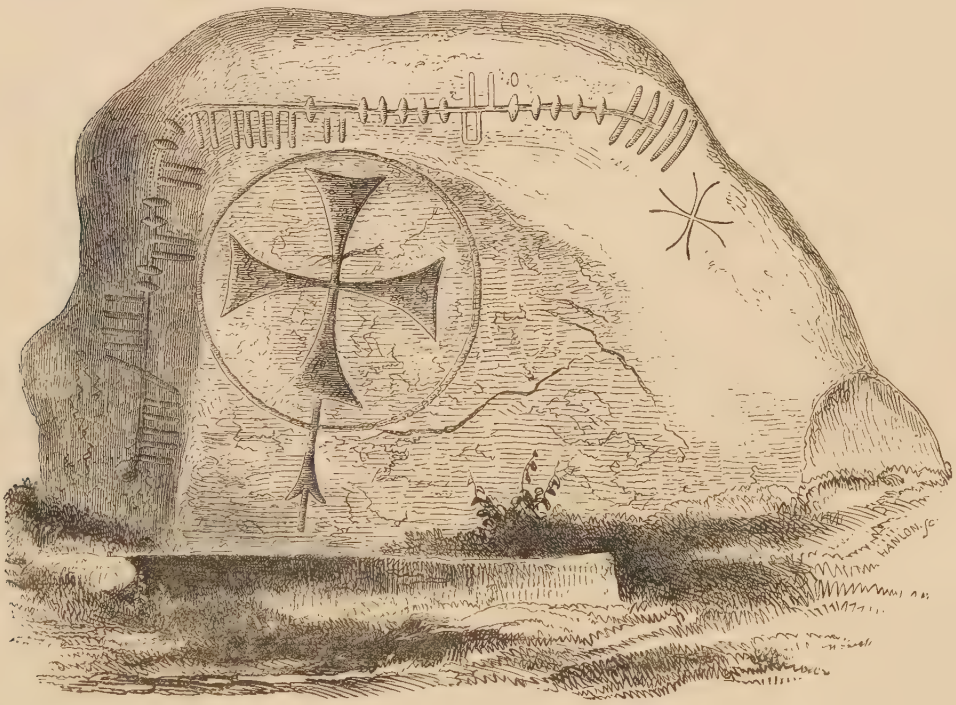
II.

ON THE OGAM MONUMENT AT KILCOLMAN. By THE RIGHT REV. CHARLES GRAVES, D.D., Bishop of Limerick.

[Read April 25, 1887.]

In a recent search, carried on with a different object in view, I happened to light upon a passage in the *Leabhar Breac* which enables me to explain an Ogam inscription upon which the ingenuity of Irish antiquaries has been, as I think, hitherto exercised with only an imperfect measure of success. It appears on a monument standing in a cealluragh, or disused burial-ground, called Kilcolman, in the townland of Maumanorig and the parish of Marhin. (Ordnance Survey of Kerry, sheet 42.) Within the enclosure are several graves, and some ruins which look like the foundations of a church. (Note A.) The monument is undoubtedly a Christian one, presenting two crosses; a large one surrounded by a circle underneath the legend, and a smaller one at the end of it. These crosses are similar to those which appear on other ancient Irish monuments, some of them bearing Ogam inscriptions, such, for instance, as those found at Aglish, Kilfountan, and Arraglen, whilst others are uninscribed. Mr. Wakeman's Paper on "Innismurray and its Antiquities" furnishes many characteristic examples. Crosses of this kind are sometimes incorrectly called Maltese crosses. They are, with more reason, called Greek. It would be safer to give them a more general designation, which would still distinguish them from Latin crosses, by calling them Oriental, as they are found in Asia Minor, Syria, and along the Nile. By whatever name we may call them, the crosses on this monument are contemporaneous with the inscription. Nothing but adherence to a preconceived theory could induce anyone to assert that they had been superadded at a later period, by way of Christianizing a monument originally pagan.

A drawing and rubbing of the monument were furnished to me by the orders of Captain Larcom, when, at his suggestion forty years ago, I undertook the attempt to decipher Ogam inscriptions; and information respecting the locality was supplied to me by the officers of the Ordnance Survey, as well as by Mr. Hitchcock, whom I employed to search for Ogams, and to make drawings and rubbings of them. I examined it myself with great care in the year 1851, noting every peculiarity observable in the several characters.



Lastly, I obtained quite recently, through the kindness of a friend, an excellent squeeze of the most doubtful part of the inscription, together with a drawing and rubbing of the whole. I saw reason to take all these pains, because there were some difficulties in the reading, and I believe it is only now that the true key to its explanation has been found.

I read the Ogam characters, which are deeply cut and arranged along an incised stem-line, as follows:—

ANMCOLOLOLNALI[L]TIR.

The Ogam graver appears to have cancelled the last L, and to have interlined that letter between the preceding A and I. (Note B.)

The group following the cancelled L presents some difficulty. Till I obtained the squeeze mentioned above, I was disposed to regard it as an H. I have now no doubt that Mr. Brash was correct in reading it as T, the first superdigit being distinct, though less deeply cut than the others, and the third being incomplete, having probably been omitted at first, and subsequently inserted above the first stroke of the I. The present condition of the surface of the stone appears to show that the first stroke of the I was never drawn upwards from the stem-line to meet what I regard as the third digit of the preceding T. If I am mistaken in this assumption, the transliteration will undergo no change of any importance. We shall merely have to read TER instead of TIR; which will be in accordance with the orthography of MSS., and of the inscription referred to in Note C. We can entertain no doubt as to the meaning of this part of the inscription. The word *n-ailithir* lies here in some form or other, however our doubts as to the mode of transliteration may be resolved. (Note C.) I made the first step in the interpretation when I showed that ANM was a formula occurring at the commencement of several Ogam inscriptions. As it enters into the proper name *Anmchad* and the noun *anmchara*, it seems most reasonable to explain it as standing for *anim* (*anima*), and thus equivalent to the common and much later form *Oroit ar anmain*, occurring in several of the sepulchral inscriptions contained in Petrie's collection. (Note D.) But how was the interpretation to be completed? The intermediate *Cololol* was enigmatical. Knowing that the church was called Kilcolman, we might have expected to find the name Colman, or some form of it, in this place. And, more than that, I always kept in view the fact mentioned by Mr. O'Connor, a member of the Ordnance Survey staff employed in Corkaguiny, that the Irish-speaking people in the neighbourhood called it Kill na g-Colman, the Church of the Colmans. This led me to suspect that the *ololol* of the inscription was a cryptic mode of signifying that three persons named Colman were commemorated by the same monument. To show that this notion was not as fanciful as it might have been thought, I now quote the following passage from the copy of the *Sanctilogium Genealogicum*, preserved in the *Leabhar Breac*:—" *Tri noeim isin oen baile .ı. a nucht*

mama. Colman ainm cech noeim ⅂ *Lugaid athair cech Colmain* .ı. *Colman m. Luigdech m. Loegaire m. Neill ix Giallach. Colman m. Luigdech m. Ængusa m. Natfraich m. Cuirc m. Luigdech. Colman m. Luigdech m. Conaill m. Briain m. Ech. Muidmedoin.*" (*Leabhar Breac*, p. 21, col. 2–3.) "Three saints in the one townland, viz. in the bosom of the maum [mountain pass]. Colman (was) the name of each saint, and Lugaid (was) the father of each Colman: viz. Colman, son of Lugaid, son of Loegaire, son of Nial of the Nine Hostages; Colman, son of Lugaid, son of Ængus, son of Natfraich, son of Corc, son of Lugaid; Colman, son of Lugaid, son of Conall, son of Brian, son of Eochaid Muidmedon." (Note E.)

The pedigrees of these three saints indicate that they may have lived at the same time, or have ministered in succession in the same church, about the end of the fifth, or the beginning of the sixth century. (Note F.)

Our late President has dealt with this inscription in different ways. (Note G.) He was at first disposed to regard the ANMCOL, with which it begins, as an anagram of Colman; but he abandoned that idea, when it had been shown that ANM was a common formula at the commencement of Ogams. In his Rhind Lectures he proposed a different solution. By regarding the third L (ᚢ) as standing for MB (ᚠ), eliminating the second OL as a meaningless syllable inserted for the purpose of disguising the name, and treating the last L as a superfetation of the same kind, he obtained the reading, ANM COLOMB NALILTER. He thus arrived at a name Colomb, representing the Colman which he expected to find. To this conclusion I cannot bring myself to assent, much as I may admire the ingenuity shown in deducing it. In the first place I can safely say that the substitution of M for the first stroke of the L is not warranted by its direction or length. The third letter of the inscription proves that the Ogam graver made a long and very oblique stroke to stand for M.

This criticism, or rather the evidence furnished by the woodcut illustrating this Paper, for the substantial accuracy of which I am responsible, is fatal to Sir Samuel Ferguson's reading. The first of the two similar subdigits, which would go to form the Ogam L, cannot be replaced by the long oblique stroke which stands for M. Like the following digit, it is short, and at right angles to the stem-stroke.

If my objections were not so decisive, it would be worth while to notice that there is a difficulty in accepting the orthography of *Colomb.* That is the nominative; and it occurs in a position where we should certainly expect a genitive. Colum would make Coluim in the genitive. Colomb, in whatever case, is a rare form, so far as I know, a *ἅπαξ λεγόμενον.* And, further, I would remark that Colman is a diminutive of Colum, rather than an equivalent. In the Catalogue of Homonymous Saints in the Book of Leinster (pp. 366–7) there is a list of 19 Colums distinguished from 209 Colmans. And, finally, I would urge, that though the artifice of interpolating meaningless syllables has been shown to exist amongst the ancient *filés*, it would be hazardous for the epigraphist to resort to such an extreme method of reducing intractable Ogams to a condition of intelligibility.

It is with regret that, in discussing this inscription, I have found myself obliged to dissent from the views stated with respect to it by our late honoured President in his Rhind Lectures—the last contribution of the fruits of his labours and studies to this department of Irish Archæology. Being himself a devoted lover of truth, he was always ready to give a patient and candid consideration to the statements and arguments of those who differed from him. It would, therefore, have been far easier for me to discuss this question with him if seated at his side in his study, than I feel it to be in this room, where all the associations of the place suggest agreement rather than difference of opinion. Here, members of this Academy, as I, who have belonged to it for half a century, am prepared to bear witness, whatever their pursuits, and however different their opinions, have co-operated with a friendly sympathy in their attempts to promote the common interests of Science, Polite Literature, and Antiquities; and these attempts, I venture to say, have not been unsuccessful.

Believing, as I do, that the inscription on this monument has the meaning, *Pro animabus*—or, *Nomina—trium Columbanorum peregrinorum*, I am prepared to admit that difficulties will be felt in showing that the words, as they stand, are in accordance with the rules of orthography and syntax. But difficulties of the same kind meet us in other Ogam inscriptions, as they do in sepulchral inscriptions in the Latin language, and of about the same age, found in Wales. If ANM stands primarily for *anima*, or *nomen*, it might

do duty as a siglum representing any number or case, and so, in like manner, N-AILTIR might represent a genitive, singular or plural. Such deviations from ordinary grammar and spelling might be expected in a cryptic writing, and it can hardly be denied that this inscription possesses that character.

I shall be ready to admit the failure of the attempt made in this Paper to identify the Maumanorig and Cill na g-Colman, in the parish of Marhin, with the maum and the church of the three Colmans mentioned in the *Leabhar Breac*, as soon as any Irish antiquary or topographer can point to a correspondence of the same names of persons and places noticeable elsewhere.

NOTES.

NOTE A.

In comparatively recent times Christian usages were kept up in the immediate neighbourhood of this monument. In the next field below the cealluragh is a well, to which people formerly came to give "Rounds." After having washed themselves in it, they performed their devotions nearer to the cealluragh and the Ogam stone.

NOTE B.

The L is inserted between A and I, according to a principle stated by me in a letter printed in the *Proceedings* of the Royal Irish Academy (2nd series, vol. ii., pt. 6, Polite Literature and Antiquities). Vowels composing a diphthong, which appears in a name or word when spelt in the manner in use in our MSS., are frequently separated by an inversion of letters. Thus, the Ogam *Curimitir* answers to *Cruimther*. It can hardly be maintained that the former is an older form. The reason for the practice is obvious enough. As all the vowels were denoted by groups of similar short strokes, varying in number from one to five, the juxta-position of two such groups might have the effect of introducing a character of doubtful power. Thus, ++++ might stand for *oi*, or *ue*, or *eu*, or *io*, and so on.

The Uraicept tells us that there were five varieties of the *Berla tobaid* (the language formed by selection or abstraction). Of these, one was the *Berla Edarscartha* (the language in which the chief letters, the vowels *a*, *o*, *u*, *e*, *i*, were separated). I suspect that this was not a dialect, but merely a pedantic mode of writing words, so as to separate the vowels which entered into the diphthongs used in the ordinary orthography.

NOTE C.

When the grave of S. Brecan, in Aran, was opened, there were found beneath a large uninscribed flag a number of rounded stones, from eight to ten inches in diameter, brought from the adjacent strand. On one of these, now in the Petrie Museum at the Royal Irish Academy, was an inscription in the Irish character:—OR AR BRAN N-AILITHER (Pray for Bran the Pilgrim). (Petrie, *Christian Inscriptions*, pt. iv., p. 20.)

NOTE D.

I have elsewhere stated that ANM might stand for AINM, *nomen*, as the phrase, "his Ogam name," is so frequently used in ancient Irish tales, where mention is made of the

setting up of sepulchral monuments. But I am not disposed to assent to the statement that this formula means *requies*, *requiesco*, or *requiescit*, as Mr. Brash has asserted. *Fanaim* or *anaim*, may mean, I stay, wait, delay, remain, or dwell. But we must have instances to show that the verb is used with the signification of the rest of the dead.

I think I have seen inscriptions in which N stands as a siglum for *nomen*, and others in which *nomen* appears to be used as equivalent to *Titulus*.

NOTE E.

Maum is a topographical name still in use in our western counties. In Kerry I have met with Maumnahaltora, Maumclossagh, Maumcoola, Mauminch, Maumnakilla, as well as Maumanorig. See Joyce, *Irish Names of Places*, p. 169.

NOTE F.

The first of these Colmans was grandson of Loegaire, King of Ireland, who died A.D. 457 (*Annals of the Four Masters*). The second was grandson of Ængus M'Natfraich, King of Munster, who was slain A.D. 489 (*Annals of the Four Masters*). The third Colman was fourth in descent from Eochoid Muidmedon, King of Ireland, who died A.D. 365 (*Annals of the Four Masters*).

It is not by any means incredible that men of royal races, and coming from distant parts of Ireland, should have lived the life of penitents or ecclesiastics in south-west Kerry. They may have gone thither as *peregrini*, expiating great crimes by years of exile and asceticism ; as S. Columbkill, the grandson of a king, banished himself, or was banished by the sentence of St. Molaisi, to Iona after the battle of Cul-dreimhne.

Of the practice of commemorating more persons than one by the same monument we have instances in the inscriptions " vii Romani," and " Or ar ii Canoin," given in Petrie's Collection, and in the Latin legend on the Killeen Cormac Ogam, " iv vere Druides," if that be the true reading. In the Martyrologies we have many instances showing that it was usual to commemorate brothers upon the same day. Thus, the Martyrology of Donegal gives five examples of the commemoration of the three sons of the same person on the same day, and four of the commemoration in like manner of seven brothers. The names of several saints are conjoined in many of the invocations in the Litany of Ængus. (*Book of Leinster*, pp. 373–4 ; *Leabhar Breac*, pp. 23–4).

NOTE G.

Sir Samuel Ferguson has expressed his views with respect to this monument, as follows :—

" The introductory *anm* occurs on so many other examples that it is easily separable from what seems the principal name beginning with *col*. I at one time thought the *anm*

and *col* were to be read both ways, outward from c, as in the puzzle-cipher of *cellach*, yielding the name Colman, to which I was encouraged by the fact of the adjoining townland, which might have included Maumanorigh as a sub-denomination, being called Kilcolman, and of there being no church or burial-ground there; and taking the name to be Colman, there seemed good reason for recognizing in the terminal groups, slightly altered, the word *ailitir*, 'pilgrim.' I had, however, to abandon this ancillary use of *anm*, on finding that combination of letters to be, as it is, a common formula prefixed to many other Ogham inscriptions, and the great difficulty of the intermediate *cololol* still remained. A closer scrutiny, however, of the text seemed to restore *Colman* in another form; for what at first sight seemed the terminal *l* before *nalilter*, having part of its first digit above the stem, might resolve itself into *mb*, making *Cololomb*, and the interjected syllable *ol* only remained to be dealt with. In this stage of the investigation my friend, Dr. Whitley Stokes, threw upon it the light which is only to be had from very rare learning and research in manuscript originals. He found in the Library of Trinity College, Dublin (H 2, 15), in the handwriting of Dudley MacFirbis, a tract called *Duil Laithne*, containing a class of words fabricated by a process called *formolad*, from ordinary Irish words, either by inserting certain meaningless syllables, or by substituting certain letters for others. The inserted syllables, of which the *Duil Laithne* affords examples, are, *osc*, *anc*, *inc*, *unc*, *nro*, *ucull*, *ros*, *es* or *os*, *air*, *aur*, *ur*, and *oll*, as *collumac*, 'power,' formed on the Irish *cumac; colluicen*, 'kitchen,' formed on the Irish *cuicenn;* just as here, by the same process, the sculptor of the Maumanorigh Ogham appears to have engraved *Cololomb* for the Irish Colomb. The *l* in the *nalilter* I would suppose to be a superfetation of the same kind, and conclude that 'Columb the pilgrim' is the person in whose epitaph these pains have been taken to disguise his name and manifest his Christian labours." (*Rhind Lectures*, pp. 38–9.)

The reader will be interested to see how Mr. Brash has dealt with this inscription. Giving first a copy of it obtained from Mr. Windele, which agrees with my own, except as regards the letter T, he afterwards presents his own reading, which very nearly coincides with that adopted by Sir Samuel Ferguson. He retains the L, which I regard as having been cancelled by a stroke parallel to the stem line, and joining the ends of the two subdigits. He agrees with me in regarding the next group as T. At this point he stops, confessing his inability to give anything like a proper rendering of this legend. However, he accepts ALILTIR as the final word, noticing that it is found as a proper name in the Annals of Tigernach, at A.D. 599, in the form of AILLILTIR. "This," he says, "is a very remarkable identification, as the name occurs but once in our indices, as far as I have been able to ascertain: of course I am not going to connect this individual with the inscription at Kilcoleman. The form Aillitir also occurs in our indices, and numerous names with the prefix Ail: originally it signified a pilgrim, and

it has come down to us from pagan times from Ail, a great upright stone or pillar, and Itrillaim, correctly Triallaim, a march, journey, walk; referring to the pilgrimages made in ancient times to the consecrated pillar-stones, holy wells, and other objects of pagan veneration, as well as to the Deiseal, or sacred walk, observed in the religious rites paid to the Ail, or stone pillar, and retained to this day in circumambulating holy wells, sunwise (Deiseal)."—(*Ogam Monuments*, p. 225.) On this I have to observe—(1) that the name given by Tigernach is Ailither, who was Abbot of Clonmacnois. His name appears in the form Elithir in the Martyrology of Donegal, at May 12th; with him it was natural to expect that Mr. Brash would not connect the inscription at Kilcolman. (2) Whether he treats it as a proper name or a common noun, Mr. Brash thinks to remove it out of the category of Christian words by providing it with a very strange etymology. It is really *Aile* + *tir*, in agreement with its meaning, *peregrinus*.

He says that there are no remains of any building in the cealluragh, which is contrary to the statements of Mr. O'Conor and Mr. Hitchcock, and my own observation. Though he notices the crosses upon the monument, he makes no remark upon this point, which is not without importance.

These crosses, and the word *Ailithir* with which the legend ends, indicate plainly enough that it belongs to the Christian period. And yet, after quoting Mr. O'Conor's description of a monument called "The Munster Man's Bed," and believed to be the grave of a warrior who was slain at the battle of Ventry, Mr. Hitchcock observes, that very little would induce him to believe that the Ogam monument at Kilcolman or Maumanorig, which is as near the scene of battle as 'Labbanirweeny,' is a monument of the same kind. "That it marks the grave of a warrior I have but little doubt—for, besides the very pagan aspect of the stone, we find engraved on it a representation of a weapon very commonly in use in those times—a spear." Mr. Hitchcock cannot have helped seeing the crosses, and must have taken it for granted that they were inscribed upon the monument to Christianize it. (Mr. Hitchcock's MS. *Notes to accompanying Sketches of Ogam Inscriptions*, p. 93, in the Library of the Royal Irish Academy.)

The practice of arranging a series of Ogam characters, not upon an edge of the stone, but along a stem-line engraved on its face, has been regarded as necessarily modern. It is certainly uncommon, for I can point to but three other instances of it amongst our Irish Ogams, namely, the Callan stone, the Kilbonane monument, and the Colman inscription at Clonmacnois, figured in Petrie's Collection. Instances, perhaps belonging to a later period, are more common amongst the Scotch Ogams. But the forms of the crosses which appear on it, and the pedigrees of the three Colmans which it commemorates, mark this monument as belonging to the first half of the sixth century.

III.

ON TWO-NOSED CATENARIES AND THEIR APPLICATION TO THE DESIGN OF SEGMENTAL ARCHES.

By T. ALEXANDER, C. E., Professor of Engineering, Trinity College, Dublin, and A. W. THOMSON, B. Sc., Assoc. Mem. Inst. C. E., Lecturer in the Glasgow and West of Scotland Technical College. Plate III.

(Communicated by the Rev. Dr. Haughton, President.)

[Read March 16, 1888.]

The Transformed Catenary is shown by Rankine (*Civil Engineering*, Art. 131) to be the form of equilibrium for an ideal linear rib or chain under the uniform-vertical-load area between itself and a horizontal straight line. This curve has received considerable attention from early times because of its importance in designing arches, and is known best, perhaps, by engineers as the equilibrium curve.

The equation to the curve, and some of its properties, are investigated by Rankine in his *Applied Mechanics*, Art. 174. He gives an approximate way of designing arch rings by its application in his *Civil Eng.*, Arts. 128, 282, and 298. In this last section is given a table for interpolating ordinates of the common catenary; and, in Art. 128, it is pointed out that a parabola, with focal distance equal to half the parameter, closely resembles the catenary. The resemblance, however, is confined to short arcs near the vertex.

It seems to have been assumed that the transformed catenary, like the common catenary and the parabola, had its curvature continuously diminishing from the vertex outwards.

In the following investigation it is shown that a very close resemblance exists between certain of these equilibrium curves and the circle; a fact important to engineers, and interesting to mathematicians. Tables of numerical results are given, illustrating the bearing of the subject, even to

those who do not care to follow the mathematical demonstrations. Supplementary Tables are added for the immediate designing of brick, sandstone, and granite arches, with circular soffits; and a few numerical examples are worked out in detail, showing their applications to varying conditions of magnitude, of proportions of parts, and of dead and moving loads.

The agreement of our designs, in every particular, with the results of the best engineering practice and the empirical formulæ founded thereon, conclusively confirms our theory. All the theories for directly designing arches hitherto put forward have disassociated the question of strength from that of stability. They can only be applied by trial and error, and have broken down under the test of practical experience.

Rankine's notation, for the most part, is employed throughout this Paper.

I. **Equation to the Curve.**—In Rankine's *Civil Engineering*, Art. 128,

$$y = \frac{m}{2}\left(\epsilon^{\frac{x}{m}} + \epsilon^{-\frac{x}{m}}\right) \tag{1}$$

is given as the equation to the common catenary of parameter m; the locus of a transformed catenary is obtained by increasing or reducing all the ordinates in a common catenary in a given ratio r; by keeping m constant, while assigning different values to r, a *family of curves*, all derived from the same common catenary, is obtained.

In one family of these curves, $r = 0$ gives the directrix XX; $r = 1$, the common catenary below, and $r = -1$ that above the directrix. For values of r greater than unity, the transformed curves are situated beyond, and for fractional values between, the pair of common catenaries.

Of the curves between the pair of common catenaries, only a sub-group lying between the pair of curves, for which $r = \frac{1}{\sqrt{3}}$ and $r = -\frac{1}{\sqrt{3}}$, possess the properties alluded to above; and for shortness, and on account of one of these properties, the Authors of this Paper have called them "*Two-nosed Catenaries.*"

From equation (1) it may be shown that all common catenaries are similar; and all transformed catenaries with a common ratio of transformation are similar to each other.

II. **Slope and Curvature.**—For a transformed catenary,

$$y = r\,\frac{m}{2}\left(\epsilon^{\frac{x}{m}} + \epsilon^{-\frac{x}{m}}\right), \tag{1}$$

$$\tan\theta = \frac{dy}{dx} = \frac{r}{2}\left(\epsilon^{\frac{x}{m}} - \epsilon^{-\frac{x}{m}}\right) = \frac{\sqrt{y^2 - y_0^2}}{m}, \tag{2}$$

$$\rho = \frac{(m^2 + y^2 - y_0^2)^{\frac{3}{2}}}{my}, \tag{3}$$

$$\sec^3\theta = (1 + \tan^2\theta)^{\frac{3}{2}} = \frac{\rho y}{m^2}, \tag{4}$$

where θ = slope of curve at any point, that is angle between tangent at point and directrix; ρ = radius of curvature at point.

For vertex

$$\rho_0 = \frac{m^2}{y_0} = \frac{m}{r}, \tag{5}$$

and at any point,

$$\sec^3\theta = \frac{\rho y}{\rho_0 y_0}, \tag{6}$$

A transformed catenary may be drawn to any degree of accuracy by plotting a number of points from equation (1), laying off the centre of curvature to each by equations (2) and (3), and from these centres drawing a succession of circular arcs. Or it may be drawn approximately without any calculation by consideration of the :—

III. **Properties of the Common Catenary.**—Fig. 1. Making r unity in the equations of last article, we have

$$y_0' = m;\ \rho' = \frac{y'^2}{m};\quad \rho_0' = m;\quad \tan\theta' = \frac{\sqrt{y'^2 - m^2}}{m};\quad \sec\theta' = \frac{y'}{m};\quad \sec^3\theta' = \frac{\rho' y'}{m^2}.$$

Since $\frac{y'}{p'} = \sec\theta'$ (fig. 1), then $p' = m$, that is, NL, the perpendicular from

foot of ordinate upon tangent, is equal to the parameter OA'; also

$$B'J \div y' = \sec\theta' = \frac{y'}{m}, \quad \therefore\ B'J = \frac{y'^2}{m} = \rho',$$

or the intercept on the normal between the common catenary and its directrix is equal to the radius of curvature; and it can be shown that the arc $A'B'$ is equal to $B'L$: all these are well known properties of the common catenary, and from them it is readily drawn for any given parameter m; thus: Draw

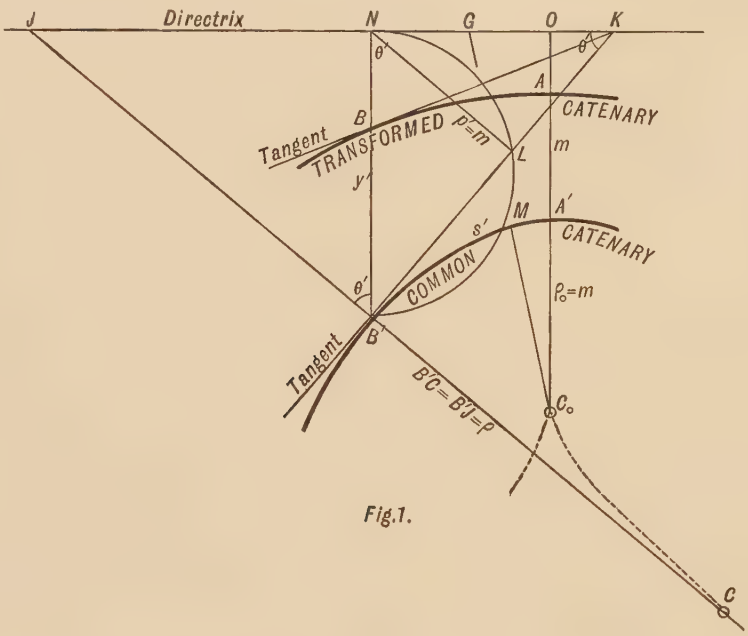


Fig.1.

the axes OX and OY; make $OA' = m$; make $A'C_0 = m$, and from C_0 as centre describe a short arc $A'M$; join C_0M and produce to meet OX at G; lay back $MC_1 = MG$, and C_1 is the centre for the next short arc, &c. The locus of the centres C_0, C_1, C_2, C, &c.; that is, the evolute of the common catenary, has a cusp at C_0 and two wings; and the common catenary has maximum curvature at its vertex. To draw a tangent at any point B'; describe a semicircle on the ordinate NB', place $NL = OA' = m$ in the semi-circle, and $B'LK$ is the tangent.

A transformed catenary of any given modulus m and ratio of transformation r can be obtained by drawing the common catenary of parameter m, as above, and then laying off $NB = r \times NB'$. A number of points, B, are readily laid off thus with proportional compasses, and joining them with a fair curve we have the transformed catenary. Join BK and we have the tangent at B.

IV. **Points of Maximum Curvature on a Transformed Catenary.**—Differentiating twice equation (3), (Art. II.), with respect to x, we have

$$\frac{d\rho}{dx} = \frac{\sqrt{m^2 + y^2 - y_0^2}\,(2y^2 - m^2 + y_0^2)}{my^2} \cdot \frac{\sqrt{y^2 - y_0^2}}{m}.$$

$$\frac{d^2\rho}{dx^2} = \frac{y^2\,[4\,(y^2 - y_0^2)\,(m^2 + y^2 - y_0^2) + (m^2 + 2y^2 - 2y_0^2)\,(2y^2 - m^2 + y_0^2)] - 2\,(m^2 + y^2 - y_0^2)\,(y^2 - y_0^2)\,(2y^2 - m^2 + y_0^2)}{m^3 y^3 \sqrt{m^2 + y^2 - y_0^2}}.$$

In order to make $\frac{d\rho}{dx} = 0$, the first factor of its numerator may not be equated to zero, since $y^2 > y_0^2$: equating the other two factors successively to zero, ρ is a maximum or minimum if $y = \sqrt{\frac{m^2 - y_0^2}{2}}$, or if $y = y_0$. Corresponding to these values of y, we find,

$$\frac{d^2\rho}{dx^2} = \frac{m^2 - 3y_0^2}{m^3} \cdot 2\sqrt{3}; \text{ and } \frac{d^2\rho}{dx^2} = \frac{3y_0^2 - m^2}{m^2 y_0}.$$

Hence if $m > y_0\sqrt{3}$, the radius of curvature will be a minimum where $y = \sqrt{\frac{m^2 - y_0^2}{2}}$, since this value of y makes the first dif. coef. zero, and the second positive. Hence also the radius of curvature at the vertex is a maximum or minimum according as $m >$ or $< y_0\sqrt{3}$; for then $y = y_0$ makes the first dif. coef. zero, and the second negative or positive.

The condition $m > y_0\sqrt{3}$ is the same as $\frac{y_0}{m} = r < \frac{1}{\sqrt{3}}$; so that of the group of transformed catenaries having their ratios of transformation less than $\pm \frac{1}{\sqrt{3}}$, each curve has a maximum radius of curvature ρ_0 at its vertex,

and a minimum radius of curvature at each of a pair of points B_1 and B_1', symmetrical about the axis, their ordinates being $y_1 = \sqrt{\frac{m^2 - y_0^2}{2}}$. This circumstance renders it desirable to give the members of this special group a *distinctive* name, which the authors have already done in Art. I. of this Paper, where the members of this group are called—

V. **Two-nosed Catenaries.**—On fig. 2, a two-nosed catenary is represented, and for the sake of clearness on such a small scale its peculiar form is somewhat exaggerated. The "*noses*" are at B_1 and B_1', and the co-ordinates, slope, &c., of a "nose" are designated by x_1, y_1, θ_1, ρ_1, &c. From Art. IV.,

$$y_1 = \sqrt{\frac{m^2 - y_0^2}{2}} = m\sqrt{\frac{1 - r^2}{2}}. \qquad (1)$$

The radius of curvature is a maximum ρ_0 at the vertex A, gradually diminishes on each side to a minimum ρ_1 at the noses B_1, B_1', beyond which it increases indefinitely; the evolute therefore has a cusp at C_0, cusps again at C_1, C_1', and two infinite branches.

VI. **Described Circle.**—Fig. 2. The normals B_1C_1 and $B_1'C_1'$ when

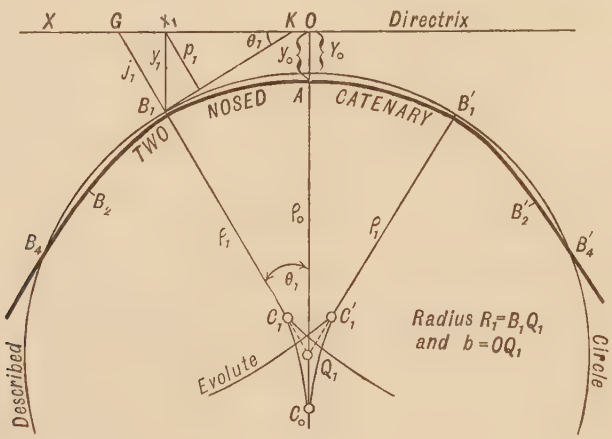


Fig. 2.

produced meet at Q_1 on the axis; so that Q_1 is the centre of the circle which touches the curve externally at the noses B_1, B_1'; this is called the "*Described*

Circle," and conversely, the curve may be said to be inscribed in this circle with regard to the given directrix. The circle is not, strictly speaking, "described," as it cuts the curve at a pair of points beyond the noses; but this can cause no ambiguity.

For the described circle, let R_1 = radius; Y = the ordinate of any point; $b = OQ_1$ = the ordinate of the centre; then the ordinate of the point on the circle nearest to the directrix, which may be called shortly its "crown," will be

$$Y_0 = b - R_1 \ldots (1), \quad \text{and } OQ_1B_1 = \theta_1 \ldots \qquad (2)$$

Corollary.—If the curve and the circle (fig. 2) be transformed towards the directrix, the curve becomes another member of the same family with a new ratio of transformation having an ellipse described upon it.

VII. **The Three-point Circle.**—Fig. 3. It is evident that there are two points, B_2 and B_2', beyond the noses, at which the curvature is the same as at A, the vertex. The co-ordinates, &c., for B_2, one of that pair, have suffix 2, thus:— x_2, y_2, θ_2, ρ_2. The circle passing through the three points, B_2, A, B_2', touches the curve internally at the vertex A, and has its centre on the axis, and may be called the "*Three-point Circle*"; this being a convenient

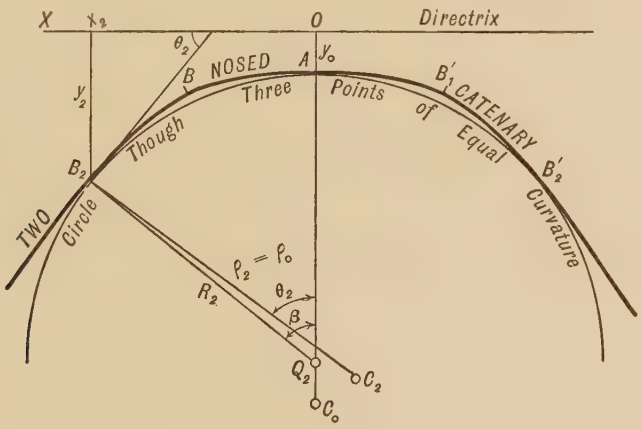


Fig.3.

abbreviation for *the circle which passes through the three points of equal curvature on a two-nosed catenary.* The radius of the three-point circle is R_2, and its

centre is at Q_2. The ordinate of A, the point on the circle nearest to the directrix, and conveniently called the "crown" of the circle, is, of course, y_0.

VIII. **Inscribed Circle.**—A circle may be drawn touching the curve internally at the vertex A, and again touching it internally at B_3 and B_3', a pair of points beyond the noses. This circle is the *Inscribed Circle;* and, conversely, the curve may be said to be described on this circle with regard to the given directrix. For B_3, one of the points of contact, the co-ordinates, &c., have suffix 3; thus:— x_3, y_3, θ_3, ρ_3; the radius of the circle is R_3, its centre is at Q_3, and the ordinate of its "crown" is y_0.

Corollary.—If the curve and its inscribed circle be drawn, and if both be transformed towards the directrix, the curve becomes another member of the same family of two-nosed catenaries, having an ellipse inscribed in it.

IX. **Properties at B_1, the Nose of a Two-nosed Catenary.**—Fig. 2. Let s be put for ratio of ordinate to radius of curvature at vertex, and

(II. (5)) $$s = \frac{y_0}{\rho_0} = y_0 \div \frac{m^2}{y_0} = r^2; \tag{1}$$

so that s is also the square of the ratio of transformation.

(V. (1)) $$y_1 = m\sqrt{\frac{1-s}{2}}; \tag{2}$$

(II. (2)) $$\tan\theta_1 = \sqrt{\frac{1-3s}{2}}; \tag{3}$$

$$\therefore \quad y_1 = \frac{m}{\sqrt{3}}\sec\theta_1, \tag{4}$$

$$p_1 = \frac{m}{\sqrt{3}}. \tag{5}$$

The result (5) is most remarkable; it is that for one family of two-nosed catenaries, at the nose of each individual member, the perpendicular from the foot of the ordinate on the tangent is *constant.* From this pro-

perty, together with the geometrical properties of the common catenary (Art. III.), some beautiful constructions can be made.

(II. (3)), (V. (1)) $$\rho_1 = \frac{(y_1^2 + 2y_1^2)^{\frac{3}{2}}}{my_1}$$

(IX. (2)) $$= m\,\frac{3\sqrt{3}}{2}(1 - s). \qquad (6)$$

Rankine's *Civ. Eng.*, Art. 131, and equation 4 :

$$x_1 = m \log_\epsilon \frac{y_1 + \sqrt{y_1^2 - y_0^2}}{y_0} \qquad (7a)$$

(IX. (1) and (2)) $$= m \log_\epsilon \frac{\sqrt{1-s} + \sqrt{1-3s}}{\sqrt{2s}}. \qquad (7b)$$

For described circle, radius $R_1 = x_1 \operatorname{cosec} \theta_1$. (8)

Ordinate of centre, $b = y_1 + R_1 \cos \theta_1$. (9)

Ordinate of crown, $Y_0 = b - R_1$. (10)

When $s = \frac{1}{49}$ sensibly, $Y_0 = 0$, and the described circle is touched by the directrix; for smaller values of s it is cut by the directrix. As s approaches zero, Y_0 is negative approaching infinity; but the ratio $Y_0 \div R_1$ approaches the limit $(\sqrt{2} - \sqrt{3}) \div \sqrt{3}$. In other words, all the two-nosed catenaries, which it is possible to inscribe in a given circle of radius R_1, must be with regard to directrices at distances from the centre of the circle not exceeding $\frac{4}{3}R_1$, nor less than $R_1 \cdot \frac{\sqrt{2}}{\sqrt{3}}$.

Lastly, $$j_1 = y_1 \sec \theta_1;$$

so that from equations (3), (4) and (6),

$$j_1 = \tfrac{1}{3}\rho_1, \qquad (11)$$

or the intercept on the normal at the nose between the curve and the directrix is one-third of the radius of curvature—a remarkable result.

The locus of the noses and the locus of the centres of curvatures at the noses are interesting curves, and may be called *friends* of the *family*. They may be plotted from Table A, and have each a horizontal asymptote.

X. **Properties at B_2.**—The point on a two-nosed catenary, where the curvature is the same as at the vertex. Fig. 3.

(II. (5)) $$\rho_2 = \rho_0 = \frac{m^2}{y_0} = m\,\frac{1}{\sqrt{s}}. \tag{1}$$

(II. (6)) $$y_2 = y_0 \sec^3\theta_2 = m\sqrt{s}\,\sec^3\theta_2. \tag{2}$$

Again, $$\rho_2 = \rho_0,$$

and from

$$\left.\begin{aligned} &(\text{II. (3) and (5)})\quad m^2 = y_0^2 + y_2^{\frac{2}{3}} y_0^{\frac{2}{3}}(y_2^{\frac{2}{3}} + y_0^{\frac{2}{3}}) \\ &(\text{X. (2)})\quad\quad\quad = y_0^2(1 + \sec^2\theta_2 + \sec^4\theta_2) \end{aligned}\right\}. \tag{3}$$

$$\frac{1}{s} = \frac{1}{r^2} = 1 + \sec^2\theta_2 + \sec^4\theta_2\,; \tag{4}$$

$$\left.\begin{aligned} \therefore\quad \sec^2\theta_2 &= \sqrt{\frac{1}{s} - \frac{3}{4}} - \frac{1}{2} \\ \tan^2\theta_2 &= \sqrt{\frac{1}{s} - \frac{3}{4}} - \frac{3}{2} \end{aligned}\right\}. \tag{5}$$

(IX. (7a)), (II. (2)) $$\begin{aligned} x_2 &= m\log_\epsilon\left(\frac{y_2}{y_0} + \frac{m}{y_0}\tan\theta_2\right) \\ &= m\log_\epsilon\left(\sec^3\theta_2 + \frac{\tan\theta_2}{\sqrt{s}}\right). \end{aligned} \tag{6}$$

Since R_2 is the radius of the circle through B_2, A and B_2'; then, by Euc. III. 35,

$$x_2^2 = (y_2 - y_0)\{2R_2 - (y_2 - y_0)\},$$

or $$R_2 = \frac{x_2^2}{2(y_2 - y_0)} + \frac{y_2 - y_0}{2}, \tag{7}$$

$$\operatorname{cosec}\beta = \frac{R_2}{x_2}\,; \quad \text{or } \tan\frac{\beta}{2} = \frac{y_2 - y_0}{x_2}. \tag{8}$$

From the tabulated results (Table A), it appears that θ_2 and β differ at most by only $2° 30'$; so we have

$$R_2(1 - \cos\beta) = y_2 - y_0 = y_0(\sec^3\theta_2 - 1);$$

or putting

$$s' = \frac{y_0}{R_2},$$

the ratio of the ordinate of the crown of the three-point circle to its radius, we have, sensibly,

$$\left.\begin{aligned} \frac{1}{s'} = \frac{R_2}{y_0} &= \sec\beta + \sec^2\beta + \sec^3\beta \\ &= \sec\theta_2 + \sec^2\theta_2 + \sec^3\theta_2 \end{aligned}\right\}. \qquad (9)$$

Table A shows that for a particular value of s, $s = \frac{1}{49}$ sensibly, the three-point circle is concentric with the described circle. The locus of B_2 is an interesting curve, which can be plotted from Table A.

XI. **Properties at B_3.**—The point on a two-nosed catenary, where the inscribed circle touches. Let 2ϕ be a subsidiary angle, whose sine is the ratio of the ordinate of A the vertex to the ordinate of B_3. The angle can be constructed by drawing a semicircle on y_3, the ordinate of B_3, and placing y_0 in the semicircle, when it will be seen that 2ϕ is $90°$ when B_3 coincides with the vertex, and approaches the limit $0°$ as B_3 recedes indefinitely from the vertex. Hence the limits of ϕ are $45°$ and $0°$, corresponding to the limiting values, the ratio of transformation $r = \frac{1}{\sqrt{3}}$ and $r = 0$.

$$\sin 2\phi = \frac{y_0}{y_3}, \text{ or } y_3 = mr \operatorname{cosec} 2\phi. \qquad (1)$$

(II. (2))
$$\tan\theta_3 = r\frac{\sqrt{y_3^2 - y_0^2}}{y_0} = r\cot 2\phi;$$

$$\therefore \quad r = \tan\theta_3 \tan 2\phi, \qquad (2)$$

$$\frac{y_3 + \sqrt{y_3^2 - y_0^2}}{y_0} = \operatorname{cosec} 2\phi + \cot 2\phi = \cot\phi.$$

(IX. (7*a*)) $$x_3 = m \log_\epsilon \cot \phi, \tag{3}$$

$$R_3 = x_3 \operatorname{cosec} \theta_3 = m \operatorname{cosec} \theta_3 \log_\epsilon \cot \phi. \tag{4}$$

(II. (4)) $$\rho_3 = \frac{m^2}{y_3} \sec^3 \theta_3. \tag{5}$$

Again, $$R_3(1 - \cos \theta_3) = y_3 - y_0 = rm(\operatorname{cosec} 2\phi - 1);$$

and, substituting r and R_3 from equations (2) and (4), and reducing,

$$\sec \theta_3 = \cot\left(\frac{\pi}{4} - \phi\right) \log_\epsilon \cot \phi - 1. \tag{6}$$

(X. (2)) $$\sec^3 \theta_3 = \operatorname{cosec} 2\phi \text{ (a close approximation).} \tag{6a}$$

By assigning to ϕ a series of values from 45° to 0°, it is easy to tabulate the corresponding values of the other quantities: to make the quantities correspond to given values of r can be done only by trials, and is difficult: only a few values are given for comparison on Table A, as in this Paper no practical use is made of the inscribed circle, the three-point circle itself being *sensibly inscribed*, and its equations more tractable.

If y_3 and y_0 be given, the solution of the inscribed circle is direct; for the values of any other two quantities given, the solution can only be made indirectly by trials: taking y_2 as a first trial value of y_3 assists greatly in making these calculations.

Results given by the equations of this article and the corresponding equations of Art. X. are approximately equal.

Table A shows that for a particular value of s, $s = \cdot 0048$ sensibly, the inscribed circle and the three-point circle coincide.

The locus of B_3 may be plotted from the Table A.

XII. **Theorem.**—If c and C be the radii of two circles, Δ the distance between their centres, and q_0 the distance intercepted between two adjacent arcs upon the line through the centres; then, with the centre

of first circle as pole, and v and θ as co-ordinates, the polar equation to the second circle is

$$v^2 - 2v\Delta \cos\theta = C^2 - \Delta^2, \tag{1}$$

where

$$\Delta = c - C + q_0. \tag{2}$$

Hence the distance between two adjacent points, one on each circle, measured normal to the first circle, is

$$q = v - c. \tag{3}$$

XIII. **Relative Position of the Described and Three-point Circles.**—Fig. 4. Let δ_0, δ_1, and δ_2 be the distance at A, B_1, and B_2; that is, at the vertex, nose, and point of equal curvature with vertex respectively, between adjacent pairs of points, one of each pair on the described and the other on the three-point circle; and measured in each instance on a radius of the three-point circle. The distance between Q_1 and Q_2, the

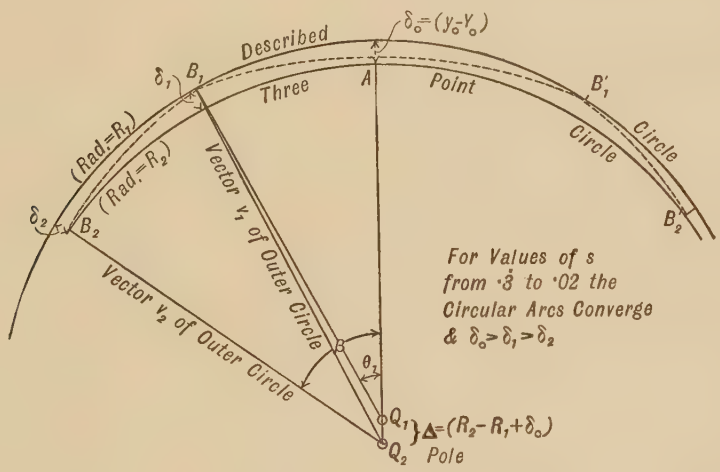


Fig. 4.

centres of these circles, is shown by Table A to be exceedingly small compared to the radii; it follows that the line from B_1, the nose to Q_2, the centre of three-point circle, sensibly makes the angle θ_1 with the

vertical. Substitute in equations of Art. XII. $c = R_2$, $C = R_1$, and $q_0 = \delta_0$; then the distance between the crowns of the two circles is

$$\delta_0 = y_0 - Y_0, \tag{1}$$

$$\Delta = R_2 - R_1 + \delta_0, \tag{2}$$

$$v_1^2 - 2v_1\Delta \cos\theta_1 = R_1^2 - \Delta^2 = v_2^2 - 2v_2\Delta \cos\beta\,; \tag{3}$$

and

$$\delta_1 = v_1 - R_2, \tag{4}$$

$$\delta_2 = v_2 - R_2. \tag{5}$$

For values of s (Art. IX.) lying between $\cdot\dot{3}$ and $\cdot 01$, very close approximate results are to be got from

$$\delta_1 = R_1 - R_2 + \Delta \cos\theta_1, \tag{4a}$$

$$\delta_2 = R_1 - R_2 + \Delta \cos\beta. \tag{5a}$$

It will be seen from Table A, that for values of s between $\cdot\dot{3}$ and $\cdot 0204$ (which includes all values on Table B), $\delta_0 > \delta_1 > \delta_2$. For any of those two-nosed catenaries, then, the arc between the pair of points B_2 and B_2' where the curvatures are the same as at the vertex, lies wholly between a pair of circles which are themselves *converging outwards from the vertex*, as on fig. 4. For any of the two-nosed catenaries on Table B, then, let a circular arc be drawn symmetrically through its vertex A, fig. 4, subtending at its own centre an angle 2β or $2\theta_2$, β and θ_2 being sensibly equal to each other, the two extreme radii being produced to cut off corresponding arcs of the curve and of the described circle; then, if this new circle through A be *concentric with the described circle*, the arc of the two-nosed catenary lies wholly within the area between those concentric circular arcs. Much more, then, does the arc of the curve lie between those of the two circular arcs, if the new circle drawn through A is not concentric with the described circle, but diverges from it outwards from the crown.

XIV. **Equilibrium of a Transformed Catenary.**—A transformed catenary is in equilibrium under a vertical load of uniform intensity w,

extending over the area between the curve and the directrix, between the curve and any other transformed catenary of the same family, or between any two members of the same family; also under a uniform load of intensity w_1, over the area between one pair of those members, together with w_2 over the area between another pair, &c.

If, for any of that group of transformed catenaries, which in this Paper are called "two-nosed," forming a boundary of the load area, we replace the *arc of the curve* from B_2 to B_2', the points of equal curvature with the vertex A, by the *arc of the three-point circle*, B_2AB_2', there will still sensibly be equilibrium. Consider the small area (fig. 3) included between those arcs: the greatest breadth of that area is the distance of the nose B_1 from the three-point circle, greatly exaggerated on fig. 3. This distance is δ_1; and it will be seen on Table A, that for values of s from $\cdot\dot{3}$ to $\cdot 0357$ (*including all the values on Table* B), the greatest value is $\delta_1 = \cdot 091$, corresponding to $R_2 = 3\cdot 48$; so that for those curves, the greatest breadth of the area included between the arcs (fig. 3) is, at most, only $\frac{1}{380}$th part of the radius.

On the other hand, when s is less than $\cdot 0357$, the noses become pronounced, and it is not self-evident that, for boundaries of loads, the curve may be replaced by the arc of the circle. This will be proved to be so in the following article:—

XV. **Theorem.**—The curve of equilibrium, for the uniform-vertical-load extending over the area between a circular soffit and a horizontal extrados, is "one-nosed," or "two-nosed," according as the ratio of depth of load at crown to the radius of the circle is greater or less than one-third; and in the second case the curve is sensibly that two-nosed catenary which has the extrados for its directrix, and the intrados for its three-point circle.

Dr. Hutton has proved, generally (see page 12 of *Theory, Practice, and Architecture of Bridges*, published in Weale's series, 1856), that, if i and $\frac{1}{\rho}$ be slope and curvature at any point of a given ideal linear rib, itself the lower boundary, or intrados of the uniform-vertical-load area which balances the rib, then the upper boundary of that load area, or extrados, is such, that

$$\frac{p\rho}{p_0\rho_0} = \sec^3 i, \qquad (1)$$

where p_0 and ρ_0 are depth of load and curvature at the highest point, where $i = 0°$, and p the depth of the load at the point in question. The equation (6) of Art. II. of this Paper, which compare, is a particular case of this. Dr. Hutton goes on to assume a variety of curves as linear rib and intrados; as, for instance, the parabola for which he finds the extrados to be another parabola. In most cases, however, he can only plot a number of points on the extrados from his equation (1) above. Some of these may be seen in the last diagram to the article "Arch," in Nichol's *Cyclopædia of Physical Sciences.*

For the semicircle, as linear rib and intrados, Dr. Hutton's diagram is as shown on fig. 5*b*. The extrados is a waved line approaching vertical asymptotes at each side, the equation (1) having now assumed the simpler form,

$$p = p_0 \sec^3 i. \tag{2}$$

Dr. Hutton's investigation for the semicircular linear rib and intrados concludes at this point. To extend it for the purposes of this Paper, take the origin at O, a height $OA = p_0 = y_0$ above the "crown" of the semicircle, and a pair of horizontal and vertical axes (figs. 5*a* and 5*b*); then, putting the ordinate of the centre $OQ = b = y_0 + c$, c being the radius, the equation to the semicircle is

$$y = b - \sqrt{c^2 - x^2}; \tag{3}$$

hence,

$$\sec i = \frac{c}{\sqrt{c^2 - x^2}}; \tag{4}$$

but,

$$p = (b - c) \sec^3 i; \tag{2a}$$

$$\therefore \quad \frac{y}{p} = \frac{b(c^2 - x^2)^{\frac{3}{2}} - (c^2 - x^2)^2}{c^3 (b - c)}. \tag{5}$$

Putting

$$z = b(c^2 - x^2)^{\frac{3}{2}} - (c^2 - x^2)^2; \tag{6}$$

then,

$$\frac{dz}{dx} = x\sqrt{c^2 - x^2}\,(4\sqrt{c^2 - x^2} - 3b), \tag{7}$$

and

$$\frac{d^2z}{dx^2} = 4(c^2 - 3x^2) - 3b\,\frac{c^2 - 2x^2}{\sqrt{c^2 - x^2}}. \tag{8}$$

Hence, $x = 0$, $x = c$, $x = \frac{1}{4}\sqrt{16c^2 - 9b^2}$, give the possible max. and min. values of the ratio $\left(\frac{y}{p}\right)$; but $x = c$ must be discarded, as p is then infinite. For $x = 0$, $\frac{d^2z}{dx^2} = c(4c - 3b)$, which is + or − according as $b >$ or $< \frac{4}{3}c$, according as $y_0 >$ or $< \frac{1}{3}c$; figs. 5*a* and 5*b* are to suit those two conditions respectively. On fig. 5*a* it is impossible that x can equal $\frac{1}{4}\sqrt{16c^2 - 9b^2}$, so that there is no further max. or min. of the ratio $y \div p$; hence the extrados on that figure continues to ascend outwards from O, as shown on the figure. On the other figure (fig. 5*b*), let

$$x_1 = \tfrac{1}{4}\sqrt{16c^2 - 9b^2}; \tag{9}$$

then, for $x = x_1$, $\frac{d^2z}{dx^2} = \frac{1}{4}(9b^2 - 16c^2)$, a negative quantity, since $b < \frac{4}{3}c$; hence, at x_1 the ratio $y \div p$ has a minimum value, and the locus of the extrados (fig. 5*b*) begins to ascend at x_1, and continues to do so outwards, indefinitely. It must therefore cross the horizontal axis at a pair of points. Let x_2 be the abscissa of M, one of the pair. It is to be got by equating the ratio in (5) to unity, when

$$c^3(b - c) = b(c^2 - x_2^2)^{\frac{3}{2}} - (c^2 - x_2^2)^2. \tag{10}$$

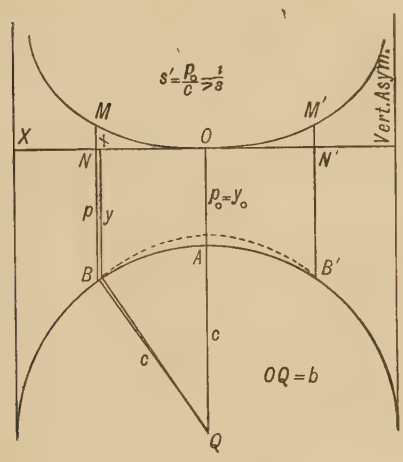


Fig. 5a.

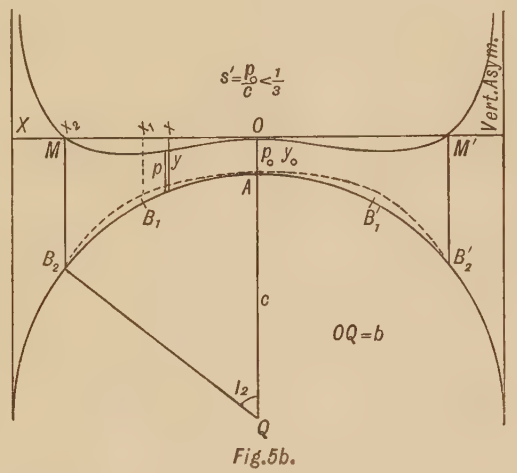


Fig. 5b.

But M is more simply defined in terms of i_2, the slope of the semicircle at B_2, the point vertically under M. For convenience, putting $s' = \frac{y_0}{c}$ the

ratio of depth of load at crown to radius, so that then $s' = (b - c) \div c$, or $(b - c) = cs'$, and $b = c(1 + s')$; $\sqrt{c^2 - x_2^2} = c \cos i_2$ may be substituted from equation (4) into equation (10); when c^4 divides out, and

$$s' = (1 + s') \cos^3 i_2 - \cos^4 i_2,$$

or,

$$\frac{1}{s'} = \sec i_2 + \sec^2 i_2 + \sec^3 i_2, \tag{11}$$

(9) becomes

$$x_1 = \frac{c}{4} \sqrt{16 - 9(1 - s')^2}. \tag{9a}$$

Taking BB', any arc of the circle (fig. 5*a*), and supposing it to consist of *two circular arcs coinciding*, the one to be considered as only the lower boundary of load or intrados, and unalterable, and the other as the ideal linear rib balanced under the load $BAB'M'OM$, and capable of altering its shape to suit alterations of that load. Suppose the portion of the load, $MOM'N'N$, to be removed so as to leave a *straight extrados*, NON', it is evident that for equilibrium the linear rib will sharpen in curvature more and more from B and B' towards the crown A, as the load taken away is more and more in excess outwards from O. The new line of stress then balanced under the uniform-vertical-load distributed over the area between the circular arc, BAB', and the straight extrados, NON', is "one-nosed"; that is, it has maximum curvature at its vertex.

On the other hand, taking the arc B_2B_2' of the circle (fig. 5*b*), and supposing it to consist of *two circular arcs coinciding*, the one to be considered as only the lower boundary of load or intrados, and unalterable, and the other as the ideal linear rib balanced under the load, in the meantime extending up to the waved extrados MOM', but capable of altering its shape to suit alterations of the load. Suppose additional load added, bringing it up to the *straight extrados*, MOM', it is evident that, to maintain the equilibrium, the linear rib must sharpen in curvature, both from A and B_2, continuously towards B_1, and from A and B_2', continuously towards B_1'; the curvature being little altered at the points B_2, A and B_2', themselves. The new line of stress will therefore be "two-nosed"; that is, it will have two points of maximum curvature at B_1 and B_1', symmetrical about the apex A, and these

will have, sensibly, the same abscissæ, x_1, as the places where the *additional load area* is deepest.

Suppose, further, that on (fig. 5*b*) a *two-nosed catenary* is drawn, having the points B_2, A and B_2', for its three points of equal curvature, that is, so that the circle is its *three-point circle;* then, $i_2 = \beta = \theta_2$, sensibly, and it will follow from equation (11) of this article, and equation (9) of Art. X., which are then indentical, that the extrados XOX is the *directrix* of that two-nosed catenary. The following Table, for a series of values of s' between its limits $\cdot\dot{3}$ and 0, shows, conclusively, that the nose of the two-nosed catenary has sensibly the same abscissa, x_1, as the nose of the altered line of stress (fig. 5*b*). (Q.E.D.)

On this Table, the first four columns are derived from Table A; the last two are calculated from equations (9*a*) and (11) of this Article, and refer to the points B_1 and B_2, a nose of the altered line of stress and one extremity, when the two hollows in Hutton's extrados are filled up level with its crown.

Two-nosed Catenaries.				Hutton's Extrados levelled.	
s	$s' = \frac{y_0}{R_2}$	x_1	θ_2	x_1	i_2
·3333	·3333	0.000	0°00′	0.000	0°00′
.2500	.2528	0.659	28 49	0.680	29 15
.1111	.1265	1.317	49 31	1.409	50 40
.0625	.0813	1.666	57 11	1.794	58 08
.0400	.0588	1.914	61 37	2.052	62 39
.0277	.0456	2.110	64 36	2.254	65 21
.0204	.0369	2.272	66 48	2.436	67 28
.0100	.0230	2.639	71 02	2.786	71 21
.0048	.0144	3.011	74 27	3.128	74 27
.0010	.0054	3.799	79 40	3.849	79 12
0	0	∞	90 00	∞	90 00

Professor Fuller's graphic construction for placing the highest balanced linear arc in the "kernel" of an arch ring already designed together with

its loading (see Jenkin's Art. on "Bridges," *Ency. Brit.*), when applied to an arch ring of uniform thickness, the depth of load at crown being considerably less than one-third of the radius of the soffit, gives always a "two-nosed" line of stress. For upon trial it will be found that the "kernel" of the arch ring, when projected out in the required manner, has a decided s-shape, and that the steepest straight line placed in it touches the lower boundary lower down, and the upper higher up. These boundaries, when projected back are *concentric circles*, and the straight line becomes the probable line of stress touching the lower circle at a pair of symmetrical points lower down, and the upper at another pair higher up, and is necessarily a "two-nosed" curve.

Rankine's graphic construction (*Civil Eng.*, figs. 119 and 120, problem IV.), when completed for a like given case of circular rib and load, indicates the same thing. The diagram of the horizontal conjugate load has a *positive area* below D, the point of rupture, and a negative area above. To have the arc from D to A, the crown, balanced under the vertical load alone, that *negative conjugate load area is left out*, and its shape indicates that the corresponding change in the linear rib, DA, will have a "nose" between D and A, or the line of stress will be "two-nosed."

XVI. **The Tables and Plates.**—On Table A are shown all the elements of the family of two-nosed catenaries with modulus *unity*, for close equidistant values of s, the square of the ratios of transformation, between the extreme limits $\cdot\dot{3}$ and 0. A few lines are given at irregular intervals where interesting changes occur. For the elements of the inscribed circle the intervals are large, as they are hard to calculate for assigned values of s, and no practical use is made of them in this Paper (see Art. XI., near its close). The Table A can be made to correspond to *any family* of two-nosed catenaries by multiplying all the *linear* quantities by the modulus, and leaving the *angular* quantities and the ratios r, s unchanged. The column i_0 gives Rankine's point of Rupture (*App. Mech.*, Art. 185, equation 13).

The Table B is derived from Table A, line by line, by dividing the linear quantities in each line by the value of R_1 in that line. Every second line on Table B, namely, those for values of s ending with 5, contains only

arithmetical averages of the lines above and below; still the quantities are correct to three decimal places. This Table is extended up and down, just so far as to suit the supplementary Tables, B1, B2, &c. Table B is, in point of fact, a series of two-nosed catenaries of different moduli, all inscribed in the circle of radius R_1 = unity, with regard to a set of parallel directrices at distances from its centre, ranging from

$$R_1 + Y_0 = 1 + \cdot 234 \text{ to } R_1 + Y_0 = 1 + \cdot 026$$

as practical limits. The theoretical limits are given (Art. IX.). The object of the Table is to secure this circle as the *upper boundary* of the "kernel" of the arch rings designed by it and the supplementary Tables B; so that when any uniform load is added, which will only be equivalent to taking a new directrix, the altered line of stress shall still *touch* the upper boundary of the "kernel," but shall not cut it, or pass above the "kernel."

The Supplementary Tables, B1, B2, &c., are derived from Table B thus:—on Table B1, $t_0 = 3\delta_0$, and $d = 2y_0 - Y_0$; on Table B3, $t_0 = 5\delta_0$, and $d = 3y_0 - 2Y_0$; on Table B4, $t_0 = 7\delta_0$ and $d = 4y_0 - 3Y_0$, &c. R and t_2 are derived as explained in the next article, so that the soffits of the arch rings shall be the three-point circles of other members of the same family of two-nosed catenaries as the lines of stress themselves in each line. Then $k = R$ vers θ_2, and $2c = 2R \sin \theta_2$. Each Table then has, for the *upper boundary* of "kernel" of arch ring (middle third, fifth, &c., of its area), a circle of radius R_1 = unity, in which the two-nosed catenary, or line of stress, is inscribed with regard to the extrados of the load as directrix. The vertex of the curve is at the lower limit of the "kernel" at crown, so that the line of stress out to θ_2 (the joint of rupture) on each side lies wholly within the "kernel" of the arch ring (Art. XIII.). At the same time, also, the soffit is the three-point circle of a two-nosed catenary of the same family as the line of stress.

The six figures, 6*a*, 6*b*, 6*c*, 6*d*, 6*e*, 6*f*, show the proportions of segmental arches between the joints of rupture, with line of stress just within a "kernel," which is the middle third of the arch ring, and the relative load area, plotted from Table B 1, for six values of s. The continual thinning of the arch ring, in passing from one to another of these figures, shows that

although in all there is *stability*, yet a point is arrived at when there will cease to be sufficient *strength;* and indicates the necessity for the other

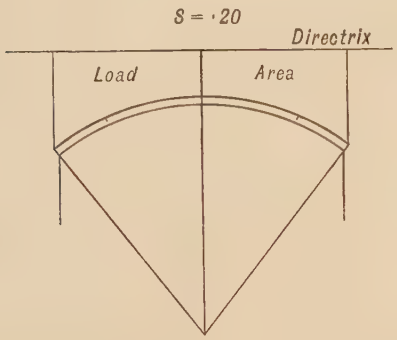


Fig.6a.

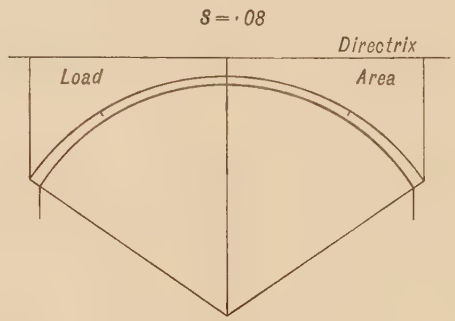


Fig.6d.

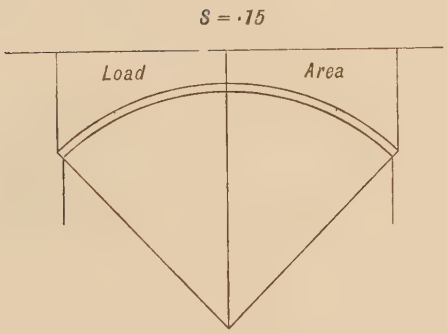


Fig.6b.

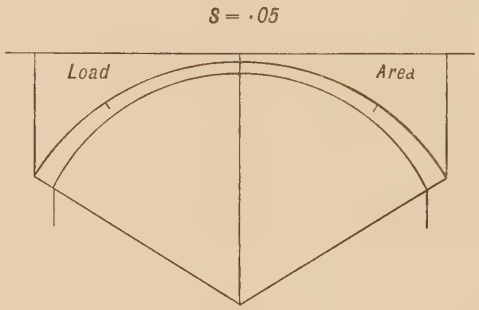


Fig.6e

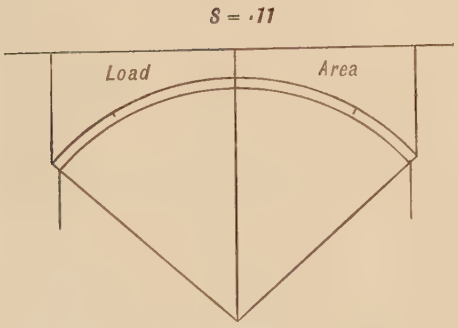


Fig.6c.

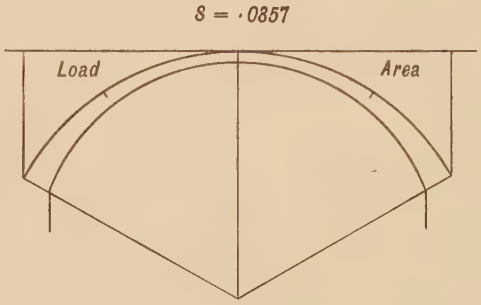


Fig.6f.

Tables, B2, B3, &c., to provide, at once, strength *and* stability for all varying proportions of the different parts.

XVII. **Thickening of Arch Ring between Crown and Haunches.**—In practice it is of great importance that the soffit of an arch ring be circular, while the equilibrium demands that the load from soffit to the extrados be duly distributed. Both these requirements are secured by adopting for soffit the three-point circle of another member of the same family of two-nosed catenaries as the line of stress itself. This circle necessarily diverges outwards from the line of stress, and gives rise to thickening of the arch ring outwards from its crown to the joints of rupture. The value of R, the radius of the soffit on any line of any of the Tables, B 1, B 2, &c., is found thus :—

Take, on Table A, the line for the corresponding value of s, in which take the quantities y_0 and δ_0; then $(y_0 + \delta_0)$, for Table B 1 (middle third), or $(y_0 + 2\delta_0)$ for Table B 3 (middle fifth), &c., will be the value of d, the depth to crown of soffit when $m = 1$. Now, on Table A, since m is unity all through, it is only necessary to find this value, d, *under the heading* y_0, and a new line is arrived at, corresponding to another member of the same family passing through crown of soffit. Take, in this new line, the quantity R_2, and it is the required radius of the soffit when m is unity. Before placing it under R in the line of the Table B 1 or B 2, &c., it must be multiplied by the value of m in that line.

To find the increased thickness, t_2, normal to the soffit at θ_2, the joint of rupture :—Substitute (in Art. XII.) $c = R$, $C = R_1 = 1$, and $q_0 = d - Y_0$; then $\Delta = R - 1 + (d - Y_0)$, and $v^2 - 2v\Delta \cos \theta_2 = 1 - \Delta^2$, which gives v. Next, $q = v - R$ is a portion of the thickness, measured normal to soffit of arch ring at θ_2, the joint of rupture; namely, from soffit to upper boundary of the "kernel." In the supplementary Table B 1, $t_2 = \frac{3}{2}q$; in Table B 3, $t_2 = \frac{5}{3}q$, &c.

Note, on these supplementary Tables, that t_2 varies from *once and a-half* to *twice* t_0, exactly as is established by the best French and British engineering practice.

XVIII. **Arch Rings of Uniform Thickness.**—Supplementary Tables for designing arch rings of uniform thickness may be prepared by modifying the Tables B 1, B 2, &c., as follows:—Alter the quantities

under the headings R, radius of soffit, and $2c$, k, span and rise of arc of soffit subtending $2\theta_2$ at its centre, thus:—Put $R = 1 - \frac{2}{3} t_0$, Table B1; $R = 1 - \frac{3}{5} t_0$, Table B3, &c.; then put $2c = 2R \sin \theta_2$, and $k = R \text{ vers } \theta_2$.

For arches designed thus with arch rings of uniform thickness the soffit is not exactly the three-point circle of another member of the same family of two-nosed catenaries as the line of stress, but is very nearly so. There will only be a *slight deficiency* of load at the haunches. For small arches, or if the moving loads be insignificant, this slight irregularity of the load will not matter, for the "kernel" of arch ring being two *concentric* circles, while the line of stress is confined between two converging circles (fig. 4), the line of stress is free to make the small necessary alteration in its form near the joint of rupture without coming out of the "kernel." For a like reason, the supplementary Tables B1, B2, &c., may be used just as they stand to design small arches with uniformly thick arch rings, by keeping to those lines where t_0 and t_2 do not differ very much, and adopting one or other, or an average of the two, for the thickness.

XIX. **Thrust on the Arch Ring.**—The horizontal thrust on arch ring due to a material of density w, taking the load as solid and extending down from directrix to the line of stress is, Rankine's, *Civ. Eng.*, Art. 131, equation 1,

$$H = wm^2. \tag{1}$$

Taking load area extending down to soffit, it is

(II. (5)) $$H = \frac{d}{y_0} wm^2 = w\rho_0 d, \tag{2}$$

where d is the depth of crown of soffit below directrix; this corresponds with theorem, Rankine's, *App. Mech.*, Art. 179. The thrust at any other point of the arch ring is

$$T = w\rho_0 d \sec \theta. \tag{3}$$

XX. **Variation of Intensity of Thrust.**—When the resultant thrust on arch ring deviates from the centre of ring by $\frac{1}{2n}$th of the thickness of the ring; then, taking the thrust as uniformly varying,

$$p\left(1 + \frac{3}{n}\right) = \text{greatest intensity of thrust,} \tag{1}$$

while $$p\left(1-\frac{3}{n}\right) = \text{least intensity of thrust,} \qquad (2)$$

where p is the average intensity. Thus, when the line of pressure is *just* within the middle third (Table B 1) of arch ring, $n = 3$, and the greatest intensity at crown joint is double the average. For middle fifth (Table B 3), n is 5, and greatest intensity of stress at crown joint is $1\frac{3}{5}$ times the average; and so on.

XXI. **Thickness of Arch Ring at Crown.**—Let t_0 = thickness of arch ring at crown; then

$$\frac{w\rho_0 d}{t_0} = \text{aver. intensity of thrust at crown joint,} \qquad (1)$$

and $$\left(1+\frac{3}{n}\right)\frac{w\rho_0 d}{t_0} = \text{max.} \quad ,, \quad ,, \quad ,, \quad ,, \qquad (2)$$

when the two-nosed catenary or line of stress is *just* confined to the middle n^{th} of the arch ring; for then the vertex of line of stress, which is the centre of stress for the crown joint, is just at the lower limit of its middle n^{th}, or deviates from the centre of the crown joint a $\frac{1}{2n}$th part of t_0, Art. XX. (1).

For ordinary sandstone the ultimate resistance to crushing is f = 576,000 lbs. per sq. ft. Basing all the Tables on *ten* as the very least factor of safety; taking the weight of sandstone at $w = 140$ lbs. per cub. ft., and supposing the whole bridge from soffit to directrix to be solid and of this weight, the last equation becomes

$$\left(1+\frac{3}{n}\right)\frac{\rho_0 d}{t_0} \leqq \frac{576000}{10 \times 140}$$
$$\leqq 411.$$

For sandstone, then,

On Table B1, $n = 3$, so that $\frac{\rho_0 d}{t_0}$ must not exceed 205.

,, ,, B3, $n = 5$, ,, $\frac{\rho_0 d}{t_0}$,, ,, ,, 257,

&c., &c.

Hence $205 \div \frac{\rho_0 d}{t_0}$ gives the greatest multiplier that may be used, line by line, on Table B1. The first column under "sandstone" on each of the eight supplementary Tables gives this maximum multiplier, while the second column gives the corresponding maximum span $2c$ between the joints of rupture, when that multiplier is used. Similar columns are given for "strong brick," and for "average granite," the quantities happening respectively to be just one-third part and double those under "sandstone." With smaller multipliers, or for a superstructure less in average weight, the factor of safety will be more than ten.

XXII. **Practical Limits of Supplementary Tables.**—The portion of the Table B to be extended on any of the supplementary Tables B1, B2, &c., is limited by practical considerations. Each is extended upwards, till with the maximum multiplier the value of t_0 becomes *one foot* for stone or *one brick* for brick-work; since practice has decided that however small an arch may be, t_0 should not be less than these quantities. Table B1 extends downwards till $t_0 = d$, which it does on the line where $s = \cdot 035$. Below this it is a physical impossibility to extend it. The other Tables would in a similar way terminate naturally, but are extended downwards only so far as the absolute value of t_2, using the maximum multiplier, remains of dimensions within which the material may be readily obtained and handled.

For any one material, inspection of these Tables shows on the highest line the *smallest span* between joints of rupture, with which it is possible to have the most perfect economy of that material; that is, the line of stress within a suitable "kernel," a factor of safety not less than ten, and a thickness in no case less than *one foot* for stone, and *one brick* for brick-work. The lowest line shows the greatest span as limited by the manner of obtaining and handling the material. Thus for segmental arches we have—

"Strong brick,"	economically	suited	for	spans, from	20 to 60	feet
"Aver. sandstone,"	,,	,,	,,	,,	50 to 140	,,
"Aver. granite,"	,,	,,	,,	,,	60 to 250	,,

agreeing with the results of practical experience.

The *variety* of Tables B1, B2, &c., is rendered necessary to give scope for all proportions of span, rise, surcharge, &c. The Tables follow each other in the order of economy of the material in the arch ring itself; but in the inverse order of economy of material in the solid backing, where the arch extends below the points of rupture. Table B1 is best suited for large spans with small surcharge: such designs are, consequently, dependent on *stability* first, *strength* following as a consequence. On Tables B7 and B8 will fall the designing of smaller arches with very heavy surcharges: such designs are then dependent on *strength* mainly, *stability* following as a consequence, since on those Tables the line of stress is sensibly along the centre of the arch ring.

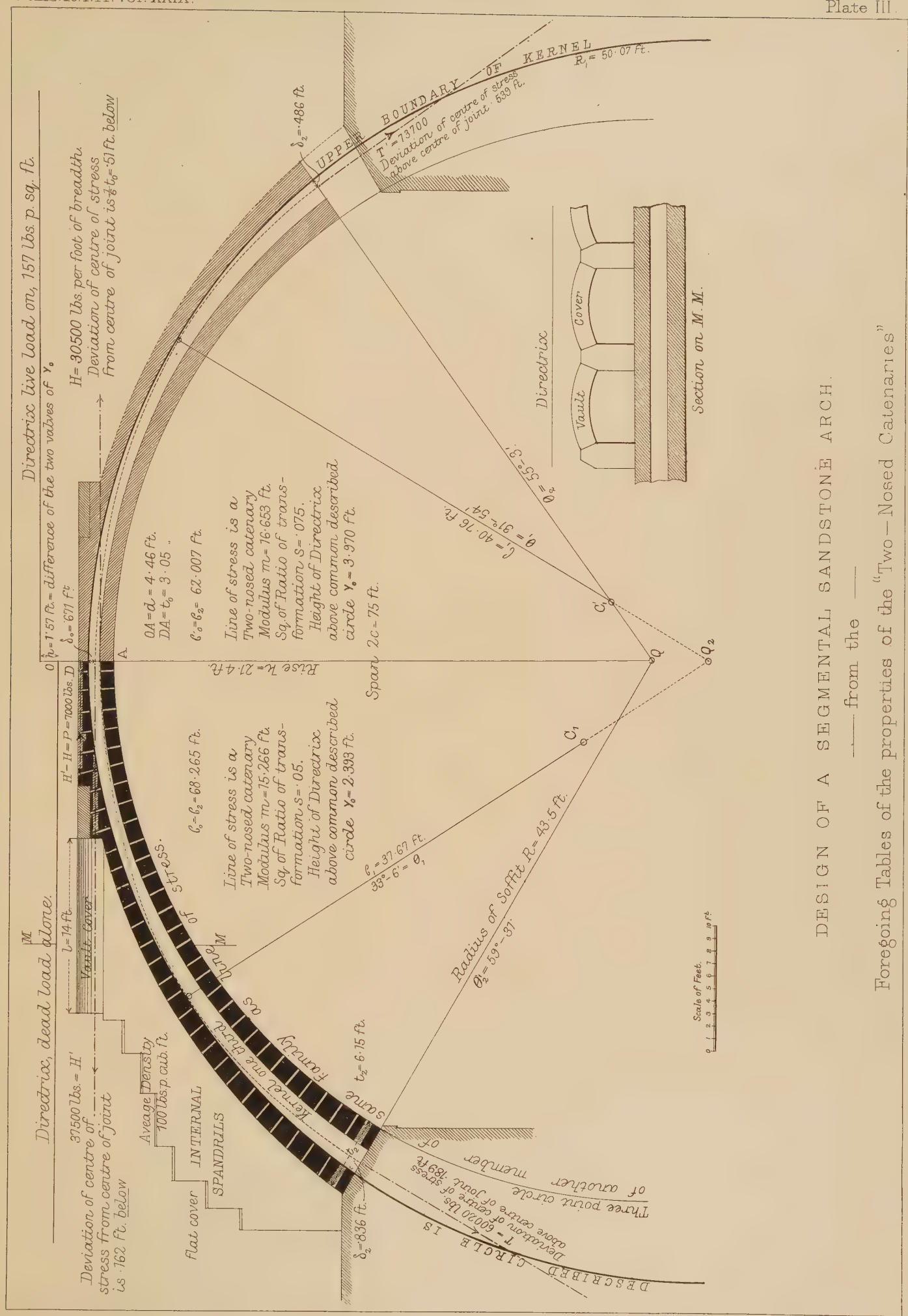
XXIII. **To strike the Line of Stress from 5 Centres** on an arch ring designed from Table B, and one of the supplementary Tables.—Prick a *pair of points* at $\frac{1}{2}\delta_0$, above and below centre of crown joint. The lower is A, the centre of stress of crown joint and vertex of line of stress; while C, below A a distance ρ_0, is its centre of curvature there. Through the upper of the *pair of points* as crown, describe an arc of a circle with radius R_1, and it is the upper boundary of the "kernel"; while B_1 and B_1', the extremities of its radii at θ_1 right and left of vertical, are the noses of the line of stress; $B_1C_1 = \rho_1 = B_1'C_1'$ laid back along those radii give the corresponding centres of curvature. Through A, the lower of the *pair of points* as a crown, draw a circular arc with radius R_2; then B_2, B_2', the extremities of its radii at the angle β right and left of the vertical, are another pair of points on line of stress near the joints of rupture, and $B_2C_2 = \rho_2 = B_2'C_2'$, laid back at the angle θ_2 with vertical, give the corresponding centres of curvature. The line of stress can be readily struck continuously in circular arcs through the five points $B_2, B_1, A, B_1' B_2'$, using C_1 and C_1' as a pair of centres; and C_2, C_0 and C_2' as first trial positions of the other three centres. This will be an exceedingly close approximation to the line of stress: the centres of stress at each joint are defined quite accurately, thus enabling the factor of safety at any joint to be calculated with precision.

The five points can be laid off with sufficient accuracy, thus:—Having

pricked the pair of points at $\frac{1}{2}\delta_0$ above and below the centre point of crown joint, of which the lower is A, and having drawn through the higher the circle of radius R_1, a line drawn from the *centre of soffit* at θ_1 to the vertical to meet this circle, determines B_1 sensibly; and from the point where another line drawn from the centre of soffit at θ_2 to the vertical meets the circle, lay back δ_2 along that line to sensibly determine B_2, which will now be on the joint of rupture. If the arch be segmental, and designed to require no solid backing, the joint of rupture will be the springing joint, and then a line drawn from B_2 normal to the springing joint enables a suitable abutment to be designed.

XXIV. **Live Load over the whole Span** (*vide* Plate III.).—After having designed an arch ring suitable for its dead load alone, by means of a suitable multiplier from a line of one of the supplementary Tables B1, B2, &c., the elements determining the line of stress are given on the same line continued on to Table B itself, which for that purpose folds out, the same multiplier for absolute values of linear quantities being used.

The alteration of this line of stress, when a given uniform load is added all over the span, has now to be considered. Reduce the live load to its equivalent height of superstructure, and add this height to the absolute value of Y_0 for a new absolute value of Y_0; or, in other words, raise the directrix through h this equivalent height. Divide this new absolute value by the multiplier previously used, and its relative value is obtained. Find a line in Table B where this new relative value of Y_0 occurs; and this *new line*, using the previous multiplier upon *linear* quantities, gives the elements of the altered line of stress when the live load is all over span. This altered line of stress *necessarily touches* the upper boundary of kernel of the arch ring with its noses just as the old line of stress did, and lies wholly within that kernel. As the new line is always a little higher up on Table B, so the noses and the B_2 points are both a little nearer the crown than before; the soffit is still *sensibly*, though not now exactly, the three-point circle of another member of the same family as this new line of stress. To prick five points on the new line of stress, having already drawn the circle forming the upper boundary of



DESIGN OF A SEGMENTAL SANDSTONE ARCH.

—— from the ——

Foregoing Tables of the properties of the "Two—Nosed Catenaries"

"kernel" of arch ring for dead load alone, as in Art. XXIII., lay down from the crown of that circle the new value of δ_0 to get A, the new apex and centre of stress at crown joint; draw from the *centre* of soffit a line at the new value of θ_1 to meet that circle at the new nose B_1, and again draw a line from centre of soffit at the new value of θ_2, and lay back the new value of δ_2 from the point where it meets that circle to obtain B_2, the centre of the new joint of rupture. A tangent line drawn from B_2 on this new joint of rupture, being more nearly horizontal, will affect the design of the abutment for heavy live loads.

The value of the horizontal thrust for dead and live loads combined is

$$H' = w\rho_0(d + h),$$

where w is the density of the material, d the depth from the directrix to crown of soffit, *i.e.* depth of dead load exactly as in finding H for dead load alone; h is the height of superstructure equivalent to the uniform live load all over span; ρ_0 is the radius of curvature of the altered line of stress at crown, and is *not* to be taken on the same line continued on Table B from the supplemental Table B 1 or B 2, &c., from which the arch ring was designed, but upon the *new* line higher up on Table B, to be found as explained above.

XXV. **Live Load over Half Span** (Plate III.).—Suppose a lever to rise vertically out of the keystone, and a horizontal force $P = H' - H$ applied to it at a suitable height, to exactly balance the two unequal horizontal thrusts. On the unloaded half we have line of stress for dead load alone, and on the loaded half for dead and live loads combined; and there will be a point of *discontinuity* at the crown. If now the point of application of P be lowered, which is equivalent to gradually removing a couple from the keystone, the arch stones will rotate; and when P has come so low as to act at some point near the centre of keystone, there shall be discontinuity no longer, but a continuous line of stress lying between the two lines of stress already mentioned, and therefore wholly within the kernel. The rotation of the arch stones near the noses will be insignificant, and

between those points an arch ring must admit of elastic motion equivalent to those rotations of the stones, in order to be suitable for irregular distribution of live load.

In actual practice P may be obtained from the resistance, frictional and otherwise, of solid backing, but most economically from the frictional resistance of vault covers supported on spandril walls built from the extrados to near the level of middle of keystone; these vault covers must therefore extend longitudinally from where they butt against the vertically-stepped backs of the highest arch stones, a distance l, such that the weight of the covers and of the permanent superstructure supported by them may give frictional resistance equal to P. Since the internal spandril walls only support these covers, they may be built of common rubble. The internal vault covers may be of long flat stones spanning the vaults, or be small relieving arches.

XXVI. **Philosophical Explanation of the Two distinct Ways in which Masonry Arches tend to Rupture** (Figs. 7, 8, 9).—It is established by experience that masonry arches tend to rupture in two distinct ways: either the crown descends, and the joints open there at the soffit, or it ascends, and the joints open at the upper side of arch ring. Moseley says "this depends upon an excess or deficiency of load at crown," but does not say what constitutes an excess or deficiency; the Roorkee Treatise on Engineering states that the two cases occurred in circular arches of which rise was less and greater than half span respectively, as if this ratio were the only one necessary condition.

Suppose an arch ring loaded that its equilibrium is *just* secured, the stability of the abutments being perfect, then *only one* line of equilibrium can be drawn in its middle third. That line for circular arches is either "one-nosed" or "two-nosed." (Art. XV.) If it be one-nosed, the crown is on the point of descending, and there are five joints of rupture tending to open inwards and outwards alternately, as shown in fig. 7. If it be two-nosed, the crown is on the point of ascending, and there are seven joints of rupture opening alternately inwards and outwards, as on fig. 8.

Generally in circular arches the value of s', the ratio of depth of load to

radius of soffit, is near $\frac{1}{10}$, and the curve is two-nosed; but so large a depth of arch ring is chosen empirically, that the highest two-nosed curve in its middle third cuts the crown joint at A near its central point; also the solid backing built up to B_2 renders the portion of the arch ring below that point part of the abutment, and throws G and G' out of consideration.

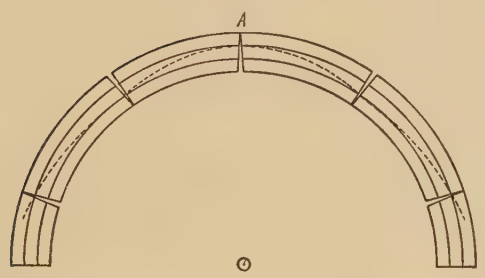


Fig.7.

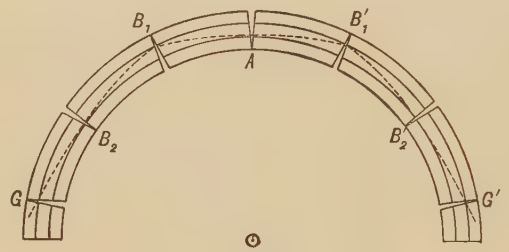


Fig.8.

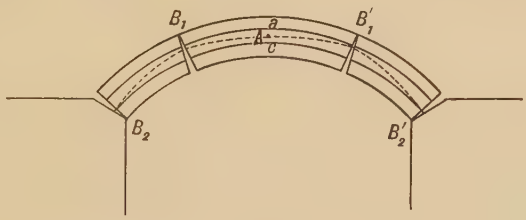


Fig.9.

The arch ring (fig. 9) is itself balanced under its load, and has an excess of stability. Suppose the equilibrium is about to be destroyed through subsidence or recession of the abutments, whether will the crown ascend or descend? If A be below the centre, the crown will ascend, as in fig. 8, and the arch ring, from B_2 to B_2', will separate into *four* parts. If A be above the centre of joint, the joint at A *will not open*, as it can only open in the direction opposite from those at B_1 and B_1'; the arch ring, from B_2 to B_2', will separate into three parts, as shown in fig. 9. It is easy to see, that if A be high in the middle third, the noses, B_1, B_1', are close together, and in this case the central portion is small.

If the abutments be forcibly thrust closer together, the crown joint will always open at the upper side of arch ring for two-nosed equilibrium curves; for then the *lowest* equilibrium curve is the line of stress, and passes through c.

Those interested in the subject may study the manner of applying the easier formulæ of this Paper to the direct design of arch rings, without Tables, in the examples set by the Authors, before Tables were prepared, to their pupils in the Imperial College of Engineering, Tokio, which are published in the Calendars of that College from 1880 to 1886.

In the future the Authors intend preparing supplementary Tables like those, B1, B2, &c., for the design of elliptic arch rings, and also to extend the use of the "Two-nosed Catenary" to the design of iron arched ribs.

In conclusion, the Authors believe that now, for the first time, it is possible to determine, with precision, the depth required for the keystone of an arch ring, by direct deduction from the principles of stability *and* strength—a problem which Rankine, in his *Civil Engineering*, Art. 290, states to be "almost impracticable from its complexity."

NUMERICAL EXAMPLES.

Example 1.—Design of a sandstone segmental arch with vertical load: span, 75 feet, and depth of surcharge at crown, about 1′ 4″. The springing to be the joint of rupture.

Here $2c = 75$, and $d - t_0 = 1{\cdot}\dot{3}$; their ratio is 56·25. We find, by trials on Table B1, that $2c \div (d - t_0) = 53$ occurs on the line where $s = {\cdot}05$, and the multiplier required on that line to make $2c$ into 75, is 50·07, about half of the max. mult. given under "sandstone"; so we shall have a factor of safety of about twice *ten*, and need not consult the other supplementary Tables, B2, B3, &c.

That line gives the following relative and absolute values:—

s	Mult.	d	t_0	t_2	R	k	$2c$
·050	50·07	·089	·061	·123	·869	·427	1·498
		4·46	3·05	6·15	43·5	21·4	75 feet.

The radius and rise of soffit are 43·5 and 21·4 feet; the thickness of arch ring at crown and springing, 3′ 0″ and 6′ 2″; the surcharge being 1·41 feet, about 1′ 4″, as required.

On the same line, continued on Table B, we have—

s	θ_2	*Mult.*	R_1	ρ_0	Y_0	δ_0	δ_2
·050	59° 31′	50·07	1	1·3634	·0478	·0204	·0167
			50·07	68·265	2·393	1·02	·836 feet.

At the crown the thrust on arch ring per foot of breadth is $H = w\rho_0 d = 140 \times 68{\cdot}265 \times 4{\cdot}46 = 42624$ lbs.; the average intensity of the stress is $42624 \div 3{\cdot}05 = 13975$; and double of this, 27950 lbs. per sq. ft., is the max. intensity, giving a factor of safety of $576000 \div 27950 = 20$.

At the springing, $T = H \sec \theta_2 = 84023$; the average stress $84023 \div 6{\cdot}15 = 13662$; and since the deviation of the centre of stress is $\frac{1}{6} t_2 - \delta_2 = 1{\cdot}025 - {\cdot}836 = {\cdot}189$ *above* the centre of the joint; then, by Art. XX., the max. intensity of stress is

$$13662 \left(1 + \frac{3}{\frac{1}{2} \cdot \frac{6{\cdot}15}{{\cdot}189}} \right) = 16181 \text{ lbs. per sq. ft.,}$$

which gives a factor of safety of $576000 \div 16181 = 35$.

Example 2.—If a live load of 220 lbs. per sq. ft. of the platform be all over the span of the bridge (Ex. 1), find the new line of stress in the arch ring and the intensities of stress at crown and springing.

The height of superstructure equivalent to this live load is $h = 220 \div 140 = 1{\cdot}571$ feet of sandstone. Here we have to find a new two-nosed catenary still inscribed in the same circle, $R_1 = 50{\cdot}07$, forming the upper boundary of "kernel" (middle third) of the arch ring, as already designed (Ex. 1), but to a directrix, h higher than before. Adding h to the old value of Y_0, we get $2{\cdot}393 + 1{\cdot}571$ or $3{\cdot}964$, which, divided by the multiplier 50·07, gives us ·0793 as a *new* relative value of Y_0, which is found in the Table B, at the line—

s	θ_2	*Mult.*	R_1	ρ_0	Y_0	δ_0	δ_2
·075	55° 3′	50·07	1	1·2384	·0793	·0134	·0097
			50·07	62.007	3·970	·671	·486 feet.

This is a new two-nosed catenary, of a different modulus and of a different family, so that the soffit already designed will not be mathematically the

three-point circle of another member of the family of this line of stress, but it will sensibly be so. The joints of rupture have gone up to 55° 3′, but this is immaterial, as the line of stress is now *closer* to the upper "kernel," and will therefore be wholly in the "kernel," down to 59° 31′, the springing joint. In designing the abutment, the tangent at the joint 53° 3′, should lie in its middle third.

At the crown, now we have the thrust, $H' = w\rho_0 (d + h) = 140 \times 62{\cdot}007$ $(4{\cdot}46 + 1{\cdot}57) = 52346$ lbs.; average intensity $52346 \div 3{\cdot}05 = 17162$; the deviation of centre of stress is $\frac{1}{6} t_0 - \delta_0 = {\cdot}509 - {\cdot}671 = {\cdot}162$ ft. *below* centre of joint, so that the max. intensity is (Art. XX.)

$$17162 \left(1 + \frac{3}{\frac{1}{2} \cdot \frac{3{\cdot}05}{{\cdot}162}}\right) = 22622 \text{ lbs. per sq. ft.,}$$

being less than max. intensity for dead load alone, because of the centre of stress being much nearer the centre of the joint. The factor of safety at crown is now twenty five.

At the springing joint, $T' = H' \sec 59° 31' = 103188$ lbs., nearly; the average intensity is $103188 \div 6{\cdot}15 = 16778$; the deviation of centre of stress is $\frac{1}{6} t_2 - \delta_2 = 1{\cdot}025 - {\cdot}486 = {\cdot}539$ ft. *above* the centre of joint. The max. intensity is

$$16778 \left(1 + \frac{3}{\frac{1}{2} \cdot \frac{6{\cdot}15}{{\cdot}539}}\right) = 25600 \text{ lbs. per sq. ft.,}$$

giving for a live load on a factor of safety at springing $576000 \div 25600 = 22$, or just about the same as at crown.

Example 3.—Let the live load (Ex. 2) cover only one-half of the span; find the horizontal thrust to be balanced by the backing of the voussoirs.

The horizontal thrust due to dead load is (Ex. 1) 42624 lbs.; and due to dead and live loads combined, it is (Ex. 2) 52346 lbs; the difference is 9722 lbs. per ft. of breadth.

Example 4.—Plate III. Suppose the arch ring, spandrils, &c., of Ex. 1

have, by means of voids in the superstructure, an average density of 100 lbs. per cub. ft. Find results corresponding to those of Ex. 1.

For stability, and to give the required surcharge, the dimensions in Ex. 1 are required, just as before, but the stresses will be altered in the ratio 140 : 100. $H = 30500$ lbs., nearly; $T = 60020$ lbs., giving factors of safety, 28 at crown and 49 at springing.

The voids in the superstructure should be so arranged that their boundary may be roughly a member of the same family as line of stress, by making the ordinates of the boundary a constant fraction of those of the soffit.

Example 5.—Plate III. Let a live load of 157 lbs. per sq. ft. of platform be over the whole span of bridge (Ex. 4); find the line of stress and the intensities of stress at crown and springing.

The equivalent height of structure is 1·57 ft., taking the new density into account; so that the solution is the same as Ex. 2, only we must alter the quantities in the ratio 140 : 100.

$H' = 37500$ lbs. nearly; $T' = 73700$ nearly, and the factors of safety are increased to 35 at crown and 31 at springing.

Example 6.—Plate III. Let the live load in Ex. 5 be only over one half of the span; find the amount of horizontal thrust to be balanced by the frictional stability of vault covers butting against the higher voussoirs. Find also the distance back to which the vault covers must extend to balance it.

The thrust is $P = H' - H = 37500 - 30500 = 7000$ lbs. per foot of breadth. If the under side of the vault covers come up to the level of the crown of the soffit, then the weight per foot of breadth of bridge on the spandrils due to the vault covers, and dead load over them alone, is $wdl = 140 \times 4{\cdot}46\,l = 624\,l$. Taking the coefficient of friction at ·7; then $\cdot 7 \times 624\,l = 7000$ or $l = 16$ ft. to the nearest foot. The voussoirs near the key-stone should have square-dressed side joints until the sum of their vertical projection is t_0, the depth of key-stone, so as to receive the horizontal longitudinal resistant thrust of the vault covers truly; and the covers are to be built with square-dressed side joints, set closely so as

to yield as little as possible. The spandril walls must be built up to level of crown of soffit, and extend back so as to give 16 feet longitudinally of vault covers; then the spandrils may step rapidly down to lower levels.

Example 7.—Design of a semicircular arch ring of common sandstone, the span to be 100 ft., and a surcharge of at least $1\frac{1}{2}$ ft. being required for the formation of the roadway, laying of gas-pipes, &c. The data are $R = 50$, and $R \div (d - t_0)$ not to be less than 33. On Table B1, the lines above that with $s = \cdot 08$ (in order to make R into 50) require a multiplier greater than the maximum given for sandstone; these lines are therefore excluded on the question of strength, while the lines below that with $s = \cdot 05$ do not give $R \div (d - t_0)$ so great as 33, and are excluded by requirements of the roadway. Those two limiting lines give—

s	*Mult.*	R	d	t_0	θ_2	Factor of Safety.
·08	53·6	50	5·9	2	54° 14′	$\frac{57 \times 10}{53 \cdot 6} = 10 \cdot 6.$
·05	57·5	50	5·1	3·5	59° 31′	$\frac{104 \times 10}{57 \cdot 5} = 18.$

The upper gives *greatest economy* of material in *arch ring*, which is only 2 ft. at crown, but *less economy* of material in superstructure, as d is larger, and also *less economy* of solid backing, which has to be built to a joint 5° higher. Hence the line midway between them would be most suitable all round. For a single arch a line a little nearer the upper may be adopted; and for a series of arches a line nearer the lower, that is, in favour of a heavier arch ring to withstand the shocks transmitted from arch to arch. The best lines then are—

	s	*Mult.*	R	d	t_0	t_2	θ_2	Factor of Safety.
Single Arch,	·07	54·526	50	5·6	2·4	4·4	55° 53′	$\frac{69 \times 10}{54 \cdot 5} = 12 \cdot 7.$
Series of Arches,	·06	55·804	50	5·3	2·9	5·4	57° 38′	$\frac{85 \times 10}{55 \cdot 8} = 15 \cdot 2.$

Compare Rankine's empirical rule, *Civ. Eng.*, Art. 290, giving

$$t_0 = \sqrt{\cdot 12 \times 50} \quad \text{and} \quad \sqrt{\cdot 17 \times 50}$$
$$= \quad 2\cdot 45 \quad ,, \quad 2\cdot 92 \text{ respectively.}$$

The solid backing must be brought up to the point where the joint at θ_2 meets the back of the arch ring, and below that joint the arch ring may be of the uniform thickness t_2. The superstructure may readily be reduced by voids and the employment of material of less density than sandstone, till the average density of the whole is a fifth less than that of sandstone, which would raise the factors of safety at crown to 16 and 19. The factors of safety at joint of rupture are even greater as the centre of stress is nearer the centre of the joint, and $t_2 \div t_1 > \sec \theta_2$. By means of the values obtained for w, ρ_0, d, δ_2, the thrust at crown and joint of rupture, and the centre of stress at joint of rupture are calculated, as in preceding example. A tangent from this last point enables a suitable abutment to be designed.

Example 8.—Design of a segmental brick arch, having the joints of rupture at the springing, a span of 25 ft., and a surcharge of $2\frac{1}{2}$ ft., slightly more or less.

Here the ratio $2c \div (d - t_0)$ is 10. On the Tables B1, B2, B3 and B4, this ratio, for the lines where the maximum span under "strong brick" exceeds our given span, is always greater than 20, 15, 12, and 11 respectively. On Table B5 we have the ratio sufficiently near to 10 on the line where $s = \cdot 14$; but the factor of safety would only be $\frac{30 \times 10}{25} = 12$ with the very strongest brick, which is rather little. On Table B6, the next following, where the line of stress is confined to the *middle thirteenth*, the given ratio brings us to the line where $s = \cdot 145$, and

s	*Mult.*	$2c$	d	t_0	t_2	R	ρ_0	θ_2	Factor of Safety.
·145	19·7	25	3·6	1·06	1·7	17·8	21·4	44° 42′	$\frac{39 \times 10}{25} = 15\cdot 6$

would suffice for the arch ring built of the strongest red bricks moulded

into the proper wedge-shaped forms. The line of stress may be exactly located, as in previous examples; but it would be sufficiently near in this case to take it as being up the very centre of the arch ring; and a line drawn normal to the springing joint from its middle point is sensibly the line of thrust on the abutment. The superstructure could easily be arranged with voids to make the average mass of the whole $w = 90$ lbs. per cubic ft., or $\frac{3}{4}$ths that of brickwork, which would increase the factor of safety to about 20. The thrusts then would be—

At Crown, $H = w\rho_0 d$ = 6934 lbs. per ft. of breadth of arch ring.

At Springing, $T = H \sec \theta_2 = 9755$ lbs. ,, ,, ,, ,,

For ordinary-shaped bricks the arch ring ought to be built of a uniform thickness, the line of stress and thrust remaining sensibly unchanged, as the extra part of the arch ring at crown will really form part of the superstructure. For a single arch, an average of t_0 and t_2 could be taken; but for a series of arches the greater had better be chosen, giving, respectively,

$$t = 1{\cdot}38 \text{ and } 1{\cdot}7 \text{ feet.}$$

Compare Rankine's empirical formula, *Civ. Eng.*, Art. 290—

$$t = \sqrt{17{\cdot}8 \times {\cdot}12} \text{ and } \sqrt{17{\cdot}8 \times {\cdot}17}$$
$$= \quad 1{\cdot}46 \quad ,, \quad 1{\cdot}74 \quad \text{respectively.}$$

Example 9.—If the bridge, left-half of Plate III., is to carry a road, approaching with a slope 1 in 100, and passing over with a curved profile sensibly circular; to find position of profile.

To make the profile sensibly a member of the same family as line of stress, it is sufficient to derive it from the soffit, which is sensibly a member of the family. Let u_0 and u_2 be depth to profile at crown and springing.

Slope of half chord of profile is 1 in 200, hence $u_2 - u_0 : 37{\cdot}5 :: 1 : 200$, and $u_2 : u_0 :: 4{\cdot}46 + 21{\cdot}4 : 4{\cdot}46$. Hence $u_0 = {\cdot}04$ ft., and $u_2 = {\cdot}23$ ft. The profile is to be below the directrix about $\frac{1}{2}$ an-inch at crown, and about 3 inches at springing. The load left out between directrix and profile increases the factors of safety as ·04 : 4·46.

[TABLES A AND B.

				DESCRIBED CIRCLE.						THREE-		
m.	s.	r.	y_0.	x_1.	y_1.	θ_1.	ρ_1.	R_1.	Y_0.	x_2.	y_2.	
						° ′						
1	$.\dot{3}$	$\sqrt{\frac{1}{3}}$	.5774	0.0000	.5774	0.00	1.7321	1.7321	+.5774	0.0000	0.5774	
1	.32	—	.5657	0.2475	.5831	8.03	1.7667	1.7672	.5657	0.3511	0.6009	1
1	.30	—	.5477	0.3977	.5916	12.36	1.8187	1.8224	.5477	0.5671	0.6382	1
	.28	—	.5292	0.5119	.6000	15.48	1.8706	1.8807	.5290	0.7336	0.6780	
	.26	—	.5099	0.6116	.6083	18.21	1.9226	1.9427	.5095	0.8808	0.7208	
	.25	$\frac{1}{2}$	.5000	0.6585	.6124	19.28	1.9485	1.9754	.4994	0.9507	0.7435	2
	.24	—	.4899	0.7041	.6164	20.31	1.9745	2.0092	.4890	1.0191	0.7671	
	.22	—	.4690	0.7932	.6245	22.24	2.0265	2.0810	.4674	1.1538	0.8174	
	.20	—	.4472	0.8814	.6325	24.06	2.0785	2.1589	.4444	1.2884	0.8727	3
	.19	—	.4359	0.9258	.6364	24.53	2.1044	2.2007	.4322	1.3567	0.9025	
	.18	—	.4243	0.9706	.6403	25.37	2.1304	2.2446	.4196	1.4260	0.9339	
	.17	—	.4123	1.0163	.6442	26.20	2.1564	2.2910	.4065	1.4969	0.9672	
	.16	—	.4000	1.0630	.6481	27.01	2.1824	2.3400	.3927	1.5696	1.0026	
	.15	—	.3873	1.1110	.6519	27.40	2.2084	2.3922	.3783	1.6447	1.0404	4
	.14	—	.3742	1.1606	.6557	28.18	2.2344	2.4477	.3631	1.7226	1.0809	
	.13	—	.3606	1.2122	.6600	28.55	2.2603	2.5075	.3471	1.8040	1.1247	
	.12	—	.3464	1.2663	.6633	29.30	2.2863	2.5719	.3300	1.8895	1.1721	
	$.\dot{1}$	$\frac{1}{3}$	.3333	1.3170	.6667	30.00	2.3094	2.6339	.3138	1.9697	1.2179	
	.11	—	.3317	1.3235	.6671	30.04	2.3123	2.6420	.3117	1.9800	1.2240	
	.10	—	.3162	1.3843	.6708	30.37	2.3383	2.7188	.2920	2.0766	1.2812	4
	.09	—	.3000	1.4498	.6745	31.08	2.3643	2.8037	.2706	2.1808	1.3450	
	.08	—	.2828	1.5211	.6782	31.39	2.3902	2.8988	.2471	2.2944	1.4170	
	.07	—	.2646	1.5999	.6819	32.09	2.4162	3.0068	.2209	2.4202	1.4997	
	.0625	$\frac{1}{4}$	.2500	1.6655	.6847	32.31	2.4357	3.0987	.1990	2.5247	1.5709	
	.06	—	.2450	1.6888	.6856	32.38	2.4422	3.1318	.1912	2.5619	1.5967	
	.05	—	.2236	1.7914	.6892	33.06	2.4682	3.2801	.1569	2.7255	1.7139	5
	.04	$\frac{1}{5}$	.2000	1.9141	.6928	33.33	2.4942	3.4628	.1157	2.9210	1.8613	
	.0357	—	.1889	1.9757	.6944	33.45	2.5053	3.5561	.0951	3.0189	1.9382	
	.03	—	.1732	2.0688	.6964	34.00	2.5201	3.6994	.0639	3.1667	2.0587	
	$.02\dot{7}$	$\frac{1}{6}$	.1667	2.1096	.6972	34.06	2.5259	3.7631	+.0503	3.2315	2.1131	
				For	a value	of s he	re, sensi	bly the	next, dir	ectrix to	uches de	scri
	.0204	$\frac{1}{7}$	.1429	2.2716	.6999	34.25	2.5451	4.0191	−.0037	3.4875	2.3381	
	.02	—	.1414	2.2821	.7000	34.26	2.5461	4.0360	−.0072	3.5041	2.3532	
	.01	$\frac{1}{10}$	.1000	2.6391	.7036	34.51	2.5721	4.6178	−.1249	4.0639	2.9107	
	.005	—	.0707	2.9907	.7053	35.04	2.5851	5.2065	−.2394	4.6099	3.5526	
	.0048	—	.0693	3.0114	.7054	35.04	2.5856	5.2410	−.2461	4.6418	3.5933	
				For	a value	of s here,	sensibly	the last,	three-p	oint and	inscribe	d c
1	.002	—	.0447	3.4519	.7064	35.11	2.5929	5.9909	−.3881	5.3190	4.5651	
1	.001	—	.0316	3.7994	.7068	35.14	2.5955	6.5872	−.4994	5.8495	5.4874	
1	.000	—	.0000	∞	.7071	35.16	2.5981	∞	− ∞	∞	∞	5

Modulus of family, or parameter of common catenary from which members are transformed, is taken as unity.

For this Table, the modulus of common Catenary from which the members are transformed is taken as unity; all quantities be determined from the Table. Intermediate values can be easily interpolated. Rankine's point of rupture for circular

E.				INSCRIBED CIRCLE.								
	β.	$\rho_0 = \rho_2$.	R_2.	x_3.	y_3.	θ_3.	ρ_3.	R_3.	$R_2 - R_1$.	δ_0.	δ_1.	δ_2.
	° ′					° ′						
o	0.00	1.7321	1.7321	0.000	0.577	0.00	1.732	1.732	.0000	.0000	.0000	.0000
8	11.28	1.7678	1.7673	—	—	—	—	—	.0001	.0000	.0000	.0000
8	18.08	1.8258	1.8228	—	—	—	—	—	.0003	.0000	.0000	.0000
8	22.57	1.8898	1.8818	—	—	—	—	—	.0010	.0001	.0001	.0001
o	26.56	1.9612	1.9446	—	—	—	—	—	.0019	.0004	.0003	.0001
9	28.44	2.0000	1.9777	1.052	0.803	32.09	2.052	1.977	.0023	.0006	.0004	.0002
3	30.26	2.0412	2.0120	—	—	—	—	—	.0029	.0009	.0006	.0004
8	33.36	2.1320	2.0846	—	—	—	—	—	.0037	.0017	.0013	.0008
1	36.33	2.2361	2.1635	—	—	—	—	—	.0046	.0029	.0022	.0014
9	37.57	2.2942	2.2057	—	—	—	—	—	.0050	.0037	.0029	.0018
6	39.20	2.3570	2.2500	—	—	—	—	—	.0054	.0047	.0037	.0023
1	40.41	2.4254	2.2965	—	—	—	—	—	.0055	.0059	.0047	.0030
6	42.00	2.5000	2.3456	—	—	—	—	—	.0055	.0073	.0059	.0039
o	43.19	2.5820	2.3975	—	—	—	—	—	.0053	.0090	.0074	.0050
4	44.37	2.6726	2.4527	—	—	—	—	—	.0050	.0111	.0091	.0063
9	45.55	2.7735	2.5116	—	—	—	—	—	.0041	.0135	.0113	.0079
4	47.13	2.8868	2.5748	—	—	—	—	—	.0029	.0165	.0139	.0100
1	48.22	3.0000	2.6353	2.127	1.418	54.02	3.481	2.628	.0013	.0195	.0167	.0122
1	48.31	3.0151	2.6429	—	—	—	—	—	.0009	.0200	.0171	.0126
9	49.51	3.1623	2.7170	—	—	—	—	—	−.0018	.0242	.0211	.0158
o	51.12	3.3333	2.7982	—	—	—	—	—	−.0055	.0294	.0259	.0199
4	52.36	3.5356	2.8879	—	—	—	—	—	−.0108	.0358	.0320	.0253
3	54.05	3.7796	2.9887	—	—	—	—	—	−.0181	.0437	.0397	.0324
1	55.14	4.0000	3.0732	2.683	1.837	61.13	4.876	3.061	−.0255	.0510	.0470	.0392
8	55.38	4.0825	3.1035	—	—	—	—	—	−.0283	.0537	.0497	.0418
1	57.20	4.4721	3.2374	—	—	—	—	—	−.0427	.0668	.0628	.0549
7	59.16	5.0000	3.3985	3.068	2.155	65.01	6.159	3.386	−.0642	.0843	.0809	.0738
6	60.11	5.2926	3.4796	—	—	—	—	—	−.0765	.0939	.0911	.0844
1	61.32	5.7733	3.6020	—	—	—	—	—	−.0974	.1093	.1073	.1026
6	62.08	6.0000	3.6557	3.364	2.411	67.26	7.331	3.643	−.1074	.1164	.1148	.1112
; d	escribed	and thre	e-point c	ircles ar	e concen	tric; Y_0 c	hanges s	ign.				
8	64.22	7.0000	3.8678	3.604	2.625	69.07	8.411	3.857	−.1512	.1465	.1473	.1494
6	64.31	7.0711	3.8816	—	—	—	—	—	−.1544	.1486	.1496	.1521
2	69.20	10.0000	4.3432	4.130	3.110	72.10	11.195	4.339	−.2746	.2249	.2337	.2582
7	74.08	14.1421	4.7926	—	—	—	—	—	−.4139	.3101	.3286	.3854
7	74.25	14.4338	4.8190	4.642	3.597	74.28	14.460	4.818	−.4220	.3154	.3344	.3924
de	; $\theta_2 = \beta$	$= \theta_3$; R_2	$= R_3$; ρ_0	$= \rho_2 = \rho_3$;	$(\theta_2 - \beta)$ an	d $(\theta_2 - \theta_3)$	change	signs.				
9	80.43	22.3607	5.3895	—	—	—	—	—	−.6013	.4328	.4628	.5659
o	86.01	31.6228	5.8637	5.674	4.606	77.45	22.731	5.806	−.7235	.5310	.5654	.6863
o	90.00	∞	∞	∞	∞	90.00	∞	∞	—	—	—	—

l angles are directly proportional to m. Any two of the above values being given or assumed, the values of the others may
inder vertical and conjugate horizontal loads has been calculated for certain values of s; it is given in the column i_0.

SUPPLEMENTARY TABLE B (1).

Arch rings, with the "Two-nosed Catenary," or Line of Stress lying within the Middle Third, and loaded from Directrix to a Circular Soffit which is the "Three-point Circle" of another member of the same family of transformed catenaries as the line of stress.

Fold out Table B, and it gives—

m the parameter of common catenary from which are transformed both line of stress and curve to which soffit is three-point circle.

R_1 the radius of circle forming upper boundary of middle third.

θ_1 the slope of that circle at B_1, where line of stress touches it with its nose.

θ_2 the angle of Rupture. Line of stress same curvature as at crown.

R_2 the radius of three-point circle of line of stress. This circle passes through B_0, the lower limit of middle third of crown joint.

δ_2 the distance from upper limit of middle third to B_2, the centre of pressure on joint of rupture.

ρ_0, ρ_1, and ρ_2 radii of curvature of line of stress at B_0, B_1, and B_2, from which line of stress may be drawn in circular arcs.

SUPPLEMENTARY TABLE B (1).

STRONG BRICK. Average weight **112** lbs. per cub. ft. Strength **154,000** lbs. per sq. ft.		**SANDSTONE.** Average weight **140** lbs. per cub. ft. Strength **576,000** lbs. per sq. ft.		**GRANITE.** Average weight **164** lbs. per cub. ft. Strength **1,350,000** lbs. per sq. ft.		Average intensity of stress at crown joint in lbs. per sq. ft.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	For reference.
Max. mult.	Max. span.	Max. multiplier to give factor of safety 10.	Max. value of span $2c$ with that multiplier.	Max. mult.	Max. span.	$\frac{\rho_0 d.}{t_0.}$	$d.$	$t_0.$	$t_2.$	$R.$	$k.$	$2c.$	$s.$
	FEET.		FEET.		FEET.								
				50	72	8.25	.141	.019	.026	.973	.325	1.451	.120
				55	81	7.47	.137	.021	.029	.970	.333	1.462	.115
				61	90	6.70	.133	.023	.033	.966	.341	1.473	.110
				68	101	6.07	.129	.025	.037	.962	.349	1.482	.105
		38	56	76	112	5.45	.125	.027	.041	.957	.357	1.491	.100
		42	63	84	126	4.94	.121	.029	.045	.952	.364	1.497	.095
		46	70	92	140	4.44	.117	.032	.049	.946	.372	1.504	.090
		51	78	102	156	4.03	.113	.034	.055	.940	.380	1.509	.085
19	29	57	86	114	172	3.62	.110	.037	.062	.933	.388	1.514	.080
21	32	63	95	126	190	3.31	.106	.040	.071	.925	.395	1.516	.075
23	35	69	105	138	210	3.00	.103	.044	.081	.917	.403	1.518	.070
25	39	77	116	154	232	2.71	.099	.048	.089	.907	.409	1.516	.065
28	43	85	128	170	256	2.42	.095	.052	.097	.896	.416	1.514	.060
31	47	94	141	–	–	2.20	.092	.056	.110	.883	.421	1.506	.055
35	52	104	155	–	–	1.98	.089	.061	.123	.869	.427	1.498	.050
38	57	–	–	–	–	1.80	.085	.067	.141	.850	.431	1.480	.045
42	62	–	–	–	–	1.62	.082	.073	.159	.831	.436	1.462	.040
46	67	–	–	–	–	1.44	.078	.078	.177	.812	.441	1.444	.035
One-third.	–	$205 \div \frac{\rho_0 d.}{t_0.}$	–	Double.	–	Directly prop. to R_1, and subject to any multiplier less than given max.							–

SUPPLEMENTARY TABLE B (2).

Arch rings, with the "Two-nosed Catenary," or Line of Stress lying within the Middle Fourth or Middle Two-eighths, and loaded from Directrix to a Circular Soffit which is the "Three-point Circle" of another member of the same family of transformed catenaries as the line of stress.

Fold out Table B, and it gives—

m the parameter of common catenary from which are transformed both line of stress and curve to which soffit is three-point circle.

R_1 the radius of circle forming upper boundary of middle fourth.

θ_1 the slope of that circle at B_1, where line of stress touches it with its nose.

θ_2 the angle of Rupture. Line of stress same curvature as at crown.

R_2 the radius of three-point circle of line of stress. This circle passes through B_0, the lower limit of middle fourth of crown joint.

δ_2 the distance from upper limit of middle fourth to B_2, the centre of pressure on joint of rupture.

ρ_0, ρ_1, and ρ_2 radii of curvature of line of stress at B_0, B_1, and B_2, from which line of stress may be drawn in circular arcs.

SUPPLEMENTARY TABLE B (2).

STRONG BRICK. Average weight **112** lbs. per cub. ft. Strength **154,000** lbs. per sq. ft.		**SANDSTONE.** Average weight **140** lbs. per cub. ft. Strength **576,000** lbs. per sq. ft.		**GRANITE.** Average weight **164** lbs. per cub. ft. Strength **1,350,000** lbs. per sq. ft.		Average intensity of stress at crown joint in lbs. per sq. ft.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	For reference.
Max. mult.	Max. span.	Max. multiplier to give factor of safety 10.	Max. value of span $2c$ with that multiplier.	Max. mult.	Max. span.	$\frac{\rho_0 d.}{t_0.}$	$d.$	$t_0.$	$t_2.$	$R.$	$k.$	$2c.$	$s.$
	FEET.		FEET.		FEET.								
				55	77	8.67	.156	.020	.029	.970	.298	1.398	.135
				62	88	7.64	.152	.022	.032	.967	.306	1.410	.130
				69	98	6.92	.148	.024	.035	.963	.313	1.420	.125
		38	54	76	108	6.21	.144	.026	.038	.959	.320	1.431	.120
		41	59	82	119	5.71	.141	.028	.043	.953	.327	1.437	.115
		45	65	90	130	5.21	.137	.030	.049	.947	.334	1.443	.110
		50	73	100	147	4.70	.134	.033	.054	.941	.341	1.449	.105
19	27	56	82	112	164	4.20	.130	.036	.060	.934	.348	1.455	.100
21	30	62	91	124	192	3.84	.127	.039	.066	.928	.355	1.460	.095
23	36	68	110	136	220	3.48	.123	.042	.072	.922	.363	1.466	.090
25	39	75	116	150	232	3.15	.120	.046	.079	.913	.369	1.466	.085
28	41	83	122	166	244	2.83	.116	.050	.087	.904	.376	1.466	.080
29	43	87	128	175	256	2.69	.113	.054	.098	.893	.381	1.462	.075
31	45	92	134	–	–	2.56	.110	.058	.110	.881	.387	1.458	.070
35	51	106	152	–	–	2.25	.107	.063	.123	.867	.391	1.449	.065
40	57	–	–	–	–	1.95	.103	.069	.137	.852	.396	1.440	.060
43	61	–	–	–	–	1.80	.101	.075	.156	.832	.398	1.419	.055
47	66	–	–	–	–	1.65	.099	.082	.175	.812	.400	1.398	.050
One-third.	–	$235 \div \frac{\rho_0 d.}{t_0.}$	–	Double.	–	Directly prop. to R_1, and subject to any multiplier less than given max.							–

SUPPLEMENTARY TABLE B (3).

Arch rings, with the "Two-nosed Catenary," or Line of Stress lying within the Middle Fifth, and loaded from Directrix to a circular Soffit which is the "Three-point Circle" of another member of the same family of transformed catenaries as the line of stress.

Fold out Table B, and it gives—

m the parameter of common catenary from which are transformed both line of stress and curve to which soffit is three-point circle.

R_1 the radius of circle forming upper boundary of middle fifth.

θ_1 the slope of that circle at B_1, where line of stress touches it with its nose.

θ_2 the angle of Rupture. Line of stress same curvature as at crown.

R_2 the radius of three-point circle of line of stress. This circle passes through B_0, the lower limit of middle fifth of crown joint.

δ_2 the distance from upper limit of middle fifth to B_2, the centre of pressure on joint of rupture.

ρ_0, ρ_1, and ρ_2 radii of curvature of line of stress at B_0, B_1, and B_2, from which line of stress may be drawn in circular arcs.

SUPPLEMENTARY TABLE B (3).

STRONG BRICK. Average weight 112 lbs. per cub. ft. Strength 154,000 lbs. per sq. ft.		SANDSTONE. Average weight 140 lbs. per cub. ft. Strength 576,000 lbs. per sq. ft.		GRANITE. Average weight 164 lbs. per cub. ft. Strength 1,350,000 lbs. per sq. ft.		Average intensity of stress at crown joint in lbs. per sq. ft.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	For reference.
Max. mult.	Max. span.	Max. multiplier to give factor of safety 10.	Max. value of span $2c$ with that multiplier.	Max. mult.	Max. span.	$\frac{\rho_0 d.}{t_0.}$	$d.$	$t_0.$	$t_2.$	$R.$	$k.$	$2c.$	$s.$
	FEET.		FEET.		FEET.								
				52	70	9.73	.169	.019	.026	.972	.273	1.351	.150
				59	80	8.77	.166	.021	.030	.968	.280	1.362	.145
				66	90	7.82	.162	.023	.034	.964	.287	1.373	.140
				73	101	7.09	.159	.025	.038	.960	.294	1.383	.135
		40	56	80	112	6.36	.155	.027	.042	.956	.302	1.394	.130
		45	63	90	126	5.76	.151	.029	.046	.951	.309	1.402	.125
17	24	50	71	100	142	5.17	.147	.032	.050	.945	.316	1.410	.120
18	26	55	79	110	158	4.71	.144	.035	.056	.939	.322	1.415	.115
20	29	61	87	122	174	4.25	.141	.038	.063	.932	.329	1.420	.110
22	32	67	95	134	190	3.87	.137	.041	.070	.925	.335	1.424	.105
24	35	73	104	146	208	3.50	.134	.045	.077	.917	.342	1.429	.100
27	39	81	116	162	232	3.20	.131	.049	.086	.908	.348	1.429	.095
30	43	89	128	178	256	2.90	.128	.053	.095	.899	.354	1.429	.090
32	46	97	139	–	–	2.66	.125	.057	.105	.888	.359	1.426	.085
35	50	106	151	–	–	2.42	.122	.062	.115	.877	.365	1.423	.080
38	54	–	–	–	–	2.22	.120	.067	.128	.862	.368	1.414	.075
42	59	–	–	–	–	2.03	.117	.073	.142	.848	.372	1.404	.070
46	64	–	–	–	–	1.87	.115	.079	.159	.831	.375	1.390	.065
50	69	–	–	–	–	1.71	.113	.086	.176	.814	.378	1.376	.060
One-third.	–	$257 \div \frac{\rho_0 d.}{t_0.}$	–	Double.	–	Directly prop. to R_1, and subject to any multiplier less than given max.							–

SUPPLEMENTARY TABLE B (4).

Arch rings, with the "Two-nosed Catenary," or Line of Stress lying within the Middle Seventh, and loaded from Directrix to a Circular Soffit which is the "Three-point Circle" of another member of the same family of transformed catenaries as the line of stress.

Fold out Table B, and it gives—

m the parameter of common catenary from which are transformed both line of stress and curve to which soffit is three-point circle.

R_1 the radius of circle forming upper boundary of middle seventh.

θ_1 the slope of that circle at B_1, where line of stress touches it with its nose.

θ_2 the angle of Rupture. Line of stress same curvature as at crown.

R_2 the radius of three-point circle of line of stress. This circle passes through B_0, the lower limit of middle seventh of crown joint.

δ_2 the distance from upper limit of middle seventh to B_2, the centre of pressure on joint of rupture.

ρ_0, ρ_1, and ρ_2 radii of curvature of line of stress at B_0, B_1, and B_2, from which line of stress may be drawn in circular arcs.

SUPPLEMENTARY TABLE B (4).

STRONG BRICK. Average weight **112** lbs. per cub. ft. Strength **154,000** lbs. per sq. ft.		**SANDSTONE.** Average weight **140** lbs. per cub. ft. Strength **576,000** lbs. per sq. ft.		**GRANITE.** Average weight **164** lbs. per cub. ft. Strength **1,350,000** lbs. per sq. ft.		Average intensity of stress at crown joint in lbs. per sq. ft.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	For reference.
Max. mult.	Max. span.	Max. multiplier to give factor of safety 10.	Max. value of span $2c$ with that multiplier.	Max. mult.	Max. span.	$\frac{\rho_0 d.}{t_0.}$	$d.$	$t_0.$	$t_2.$	$R.$	$k.$	$2c.$	$s.$
	FEET.		FEET.		FEET.								
				52	68	11.10	.188	.018	.026	.973	.240	1.280	.170
				59	75	9.98	.184	.020	.029	.969	.247	1.293	.165
				66	83	8.85	.180	.022	.032	.965	.255	1.307	.160
				73	96	7.97	.177	.024	.035	.961	.262	1.318	.155
		41	55	82	110	7.10	.173	.026	.039	.957	.269	1.330	.150
		45	60	90	120	6.42	.170	.029	.044	.952	.275	1.338	.145
		50	67	100	134	5.74	.166	.032	.049	.946	.282	1.347	.140
		55	75	110	150	5.22	.163	.035	.055	.940	.288	1.354	.135
20	28	61	83	122	166	4.70	.160	.038	.061	.934	.295	1.362	.130
22	31	68	93	136	186	4.28	.157	.041	.068	.927	.301	1.366	.125
25	34	75	103	150	206	3.86	.154	.045	.075	.919	.307	1.371	.120
27	37	82	113	164	226	3.53	.151	.049	.083	.910	.312	1.372	.115
30	41	90	124	180	248	3.20	.148	.053	.091	.901	.318	1.373	.110
33	45	99	136	198	272	2.93	.146	.057	.101	.890	.323	1.371	.105
36	49	108	148	–	–	2.67	.143	.062	.112	.879	.328	1.369	.100
39	53	118	161	–	–	2.45	.141	.067	.123	.867	.332	1.364	.095
43	56	–	–	–	–	2.24	.138	.073	.135	.854	.336	1.358	.090
47	61	–	–	–	–	2.07	.137	.080	.150	.839	.340	1.352	.085
51	68	–	–	–	–	1.90	.135	.087	.166	.823	.345	1.346	.080
One-third.	–	$288 \div \frac{\rho_0 d.}{t_0.}$	–	Double.	–	Directly prop. to R_1, and subject to any multiplier less than given max.							–

SUPPLEMENTARY TABLE B (5).

Arch rings, with the "Two-nosed Catenary," or Line of Stress lying within the Middle Ninth, and loaded from Directrix to a Circular Soffit which is the "Three-point Circle" of another member of the same family of transformed catenaries as the line of stress.

Fold out Table B, and it gives—

m the parameter of common catenary from which are transformed both line of stress and curve to which soffit is three-point circle.

R_1 the radius of circle forming upper boundary of middle ninth.

θ_1 the slope of that circle at B_1, where line of stress touches it with its nose.

θ_2 the angle of Rupture. Line of stress same curvature as at crown.

R_2 the radius of three-point circle of line of stress. This circle passes through B_0, the lower limit of middle ninth of crown joint.

δ_2 the distance from upper limit of middle ninth to B_2, the centre of pressure on joint of rupture.

ρ_0, ρ_1, and ρ_2 radii of curvature of line of stress at B_0, B_1, and B_2, from which line of stress may be drawn in circular arcs.

SUPPLEMENTARY TABLE B (5).

STRONG BRICK. Average weight 112 lbs. per cub. ft. Strength 154,000 lbs. per sq. ft.		SANDSTONE. Average weight 140 lbs. per cub. ft. Strength 576,000 lbs. per sq. ft.		GRANITE. Average weight 164 lbs. per cub. ft. Strength 1,350,000 lbs. per sq. ft.		Average intensity of stress at crown joint in lbs. per sq. ft.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	For reference.
Max. mult.	Max. span.	Max. multiplier to give factor of safety 10.	Max. value of span $2c$ with that multiplier.	Max. mult.	Max. span.	$\frac{p_0 d}{t_0}$.	d.	t_0.	t_2.	R.	k.	$2c$.	s.
	FEET.		FEET.		FEET.								
				58	72	10.90	.197	.019	.028	.964	.223	1.234	.180
				64	80	9.82	.194	.021	.031	.963	.230	1.250	.175
				70	88	8.74	.190	.023	.034	.962	.238	1.266	.170
		40	50	80	101	7.86	.187	.025	.037	.958	.245	1.278	.165
		44	57	88	114	6.98	.183	.028	.041	.953	.252	1.290	.160
		49	64	98	129	6.30	.180	.031	.047	.948	.258	1.299	.155
		55	72	110	144	5.62	.177	.034	.054	.942	.265	1.309	.150
20	27	61	80	122	161	5.09	.174	.037	.060	.936	.269	1.316	.145
22	30	67	89	134	178	4.57	.171	.041	.067	.929	.274	1.323	.140
25	33	75	99	150	199	4.14	.168	.045	.073	.921	.281	1.327	.135
27	37	83	110	166	220	3.72	.165	.049	.080	.913	.288	1.331	.130
30	40	91	121	182	242	3.41	.162	.053	.089	.903	.293	1.331	.125
33	44	99	132	198	264	3.10	.160	.058	.098	.893	.298	1.332	.120
36	48	108	145	–	–	2.86	.158	.063	.108	.882	.302	1.334	.115
39	52	118	157	–	–	2.62	.156	.068	.119	.870	.307	1.326	.110
42	56	–	–	–	–	2.41	.154	.074	.131	.857	.311	1.320	.105
46	61	–	–	–	–	2.21	.152	.080	.144	.844	.315	1.315	.100
50	66	–	–	–	–	2.03	.151	.087	.158	.828	.317	1.303	.095
55	71	–	–	–	–	1.86	.149	.095	.173	.812	.320	1.291	.090
One-third.	–	$308 \div \frac{p_0 d}{t_0}$.	–	Double.	–	Directly prop. to R_1, and subject to any multiplier less than given max.							–

SUPPLEMENTARY TABLE B (6).

Arch rings, with the "Two-nosed Catenary," or Line of Stress lying within the Middle Thirteenth, and loaded from Directrix to a Circular Soffit which is the "Three-point Circle" of another member of the same family of transformed catenaries as the line of stress.

Fold out Table B, and it gives—

m the parameter of common catenary from which are transformed both line of stress and curve to which soffit is three-point circle.

R_1 radius of circle forming upper boundary of middle thirteenth.

θ_1 the slope of that circle at B_1, where line of stress touches it with its nose.

θ_2 the angle of Rupture. Line of stress same curvature as at crown.

R_2 the radius of three-point circle of line of stress. This circle passes through B_0, the lower limit of middle thirteenth of crown joint.

δ_2 the distance from upper limit of middle thirteenth to B_2, the centre of pressure on joint of rupture.

ρ_0, ρ_1, and ρ_2 radii of curvature of line of stress at B_0, B_1, and B_2, from which line of stress may be drawn in circular arcs.

SUPPLEMENTARY TABLE B (6).

STRONG BRICK. Average weight **112** lbs. per cub. ft. Strength **154,000** lbs. per sq. ft.		**SANDSTONE.** Average weight **140** lbs. per cub. ft. Strength **576,000** lbs. per sq. ft.		**GRANITE.** Average weight **164** lbs. per cub. ft. Strength **1,350,000** lbs. per sq. ft.		Average intensity of stress at crown joint in lbs. per sq. ft.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	For reference.
Max. mult.	Max. span.	Max. multiplier to give factor of safety 10.	Max. value of span $2c$ with that multiplier.	Max. mult.	Max. span.	$\frac{\rho_0 d.}{t_0.}$	$d.$	$t_0.$	$t_2.$	$R.$	$k.$	$2c.$	$s.$
	FEET.		FEET.		FEET.								
				52	60	13.20	.216	.017	.025	.972	.194	1.166	.200
				60	70	11.53	.212	.019	.028	.968	.201	1.181	.195
				68	81	9.86	.208	.022	.031	.965	.208	1.197	.190
		39	47	78	94	8.72	.205	.025	.035	.961	.214	1.211	.185
		44	54	88	108	7.58	.202	.028	.039	.957	.221	1.225	.180
		49	61	99	122	6.84	.199	.031	.045	.951	.227	1.234	.175
		55	68	110	136	6.10	.196	.034	.051	.945	.233	1.244	.170
20	25	60	75	121	151	5.59	.193	.037	.056	.937	.239	1.251	.165
22	28	66	83	132	166	5.08	.190	.040	.062	.930	.246	1.259	.160
25	31	74	94	149	188	4.57	.188	.044	.070	.922	.251	1.264	.155
28	35	83	105	166	210	4.07	.185	.049	.078	.913	.257	1.269	.150
31	39	92	116	184	233	3.70	.182	.054	.087	.903	.261	1.270	.145
34	43	101	128	202	256	3.33	.180	.059	.096	.893	.266	1.272	.140
37	47	111	140	–	–	3.05	.178	.064	.106	.882	.270	1.270	.135
40	51	121	153	–	–	2.78	.176	.070	.117	.870	.275	1.268	.130
44	55	–	–	–	–	2.56	.175	.076	.129	.857	.278	1.264	.125
48	60	–	–	–	–	2.34	.173	.083	.141	.844	.282	1.259	.120
54	67	–	–	–	–	2.10	.172	.091	.154	.829	.285	1.249	.115
60	74	–	–	–	–	1.87	.170	.099	.168	.813	.287	1.239	.110
One-third.	–	$337 \div \frac{\rho_0 d.}{t_0.}$	–	Double.	–	Directly prop. to R_1, and subject to any multiplier less than given max.							–

SUPPLEMENTARY TABLE B (7).

Arch rings, with the "Two-nosed Catenary," or Line of Stress lying within the Middle Twenty-first, and loaded from Directrix to a Circular Soffit which is the "Three-point Circle" of another member of the same family of transformed catenaries as the line of stress.

Fold out Table B, and it gives—

m the parameter of common catenary from which are transformed both line of stress and curve to which soffit is three-point circle.

R_1 the radius of circle forming upper boundary of middle twenty-first.

θ_1 the slope of that circle at B_1, where line of stress touches it with its nose.

θ_2 the angle of Rupture. Line of stress same curvature as at crown.

R_2 the radius of three-point circle of line of stress. This circle passes through B_0, the lower limit of middle twenty-first of crown joint.

δ_2 the distance from upper limit of middle twenty-first to B_2, the centre of pressure on joint of rupture.

ρ_0, ρ_1, and ρ_2 radii of curvature of line of stress at B_0, B_1, and B_2, from which line of stress may be drawn in circular arcs.

SUPPLEMENTARY TABLE B (7).

STRONG BRICK. Average weight **112** lbs. per cub. ft. Strength **154,000** lbs. per sq. ft.		**SANDSTONE.** Average weight **140** lbs. per cub. ft. Strength **576,000** lbs. per sq. ft.		**GRANITE.** Average weight **164** lbs. per cub. ft. Strength **1,350,000** lbs. per sq. ft.		Average intensity of stress at crown joint in lbs. per sq. ft.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	For reference.
Max. mult.	Max. span.	Max. multiplier to give factor of safety 10.	Max. value of span $2c$ with that multiplier.	Max. mult.	Max. span.	$\frac{\rho_0 d.}{t_0.}$	$d.$	$t_0.$	$t_2.$	$R.$	$k.$	$2c.$	$s.$
	FEET.		FEET.		FEET.								
				52	56	14.00	.232	.018	.019	.974	.165	1.082	.220
				61	67	12.15	.229	.020	.024	.969	.171	1.098	.215
				70	78	10.30	.226	.023	.029	.965	.178	1.115	.210
		39	44	79	89	9.26	.223	.021	.033	.960	.184	1.130	.205
		44	50	88	100	8.23	.220	.028	.037	.955	.191	1.146	.200
		50	57	100	115	7.30	.218	.031	.044	.948	.196	1.156	.195
19	22	56	65	112	130	6.38	.215	.035	.051	.941	.202	1.167	.190
21	25	63	74	127	149	5.71	.213	.039	.057	.934	.208	1.176	.185
24	28	71	84	142	168	5.04	.210	.044	.064	.926	.214	1.185	.180
27	32	80	95	160	190	4.55	.208	.049	.072	.917	.219	1.189	.175
30	36	89	106	178	212	4.06	.205	.054	.080	.907	.224	1.194	.170
33	39	99	118	198	237	3.68	.203	.059	.090	.896	.229	1.196	.165
36	43	109	131	218	262	3.31	.202	.065	.100	.885	.234	1.198	.160
40	48	120	144	–	–	3.02	.201	.072	.112	.873	.238	1.196	.155
44	53	132	158	–	–	2.73	.200	.079	.124	.860	.242	1.195	.150
48	57	–	–	–	–	2.50	.199	.087	.137	.845	.244	1.187	.145
53	62	–	–	–	–	2.28	.198	.095	.150	.829	.247	1.180	.140
One-third.	–	$360 \div \frac{\rho_0 d.}{t_0.}$	–	Double.	–	Directly prop. to R_1, and subject to any multiplier less than given max.							–

SUPPLEMENTARY TABLE B (8).

Arch rings, with the "Two-nosed Catenary," or Line of Stress lying within the Middle Thirty-first, and loaded from Directrix to a Circular Soffit which is the "Three-point Circle" of another member of the same family of transformed catenaries as the line of stress.

Fold out Table B, and it gives—

m the parameter of common catenary from which are transformed both line of stress and curve to which soffit is three-point circle.

R_1 the radius of circle forming upper boundary of middle thirty-first.

θ_1 the slope of that circle at B_1, where line of stress touches it with its nose.

θ_2 the angle of Rupture. Line of stress same curvature as at crown.

R_2 the radius of three-point circle of line of stress. This circle passes through B_0, the lower limit of middle thirty-first of crown joint.

δ_2 the distance from upper limit of middle thirty-first to B_2, the centre of pressure on joint of rupture.

ρ_0, ρ_1, and ρ_2 radii of curvature of line of stress at B_0, B_1, and B_2, from which line of stress may be drawn in circular arcs.

SUPPLEMENTARY TABLE B (8).

STRONG BRICK. Average weight 112 lbs. per cub. ft. Strength 154,000 lbs. per sq. ft.		SANDSTONE. Average weight 140 lbs. per cub. ft. Strength 576,000 lbs. per sq. ft.		GRANITE. Average weight 164 lbs. per cub. ft. Strength 1,350,000 lbs. per sq. ft.		Average intensity of stress at crown joint in lbs. per sq. ft.	Depth of load from extrados (directrix) to crown of soffit.	Thickness of arch-ring normal to soffit at crown.	Thickness of arch-ring normal to soffit at θ_2 joint of rupture.	Radius of the soffit.	Rise of segment of soffit subtending $2\theta_2$.	Span of segment of soffit subtending $2\theta_2$.	For reference.
Max. mult.	Max. span.	Max. multiplier to give factor of safety 10.	Max. value of span $2c$ with that multiplier.	Max. mult.	Max. span.	$\frac{\rho_0 d.}{t_0}$.	d.	t_0.	t_2.	R.	k.	$2c$.	s.
	FEET.		FEET.		FEET.								
				58	60	13.00	.243	.021	.023	.970	.149	1.034	.230
				68	72	11.30	.239	.023	.025	.964	.155	1.050	.225
		39	42	78	84	9.63	.235	.025	.028	.959	.161	1.066	.220
		44	48	88	96	8.65	.233	.028	.035	.953	.167	1.081	.215
		49	54	98	108	7.67	.231	.031	.042	.948	.174	1.096	.210
19	21	57	63	114	127	6.70	.229	.036	.050	.940	.180	1.107	.205
22	24	65	73	130	146	5.74	.227	.041	.057	.932	.186	1.118	.200
25	28	74	83	148	167	5.11	.225	.046	.065	.922	.191	1.124	.195
28	32	83	94	166	188	4.49	.223	.052	.074	.912	.196	1.131	.190
31	36	94	107	188	214	4.04	.222	.058	.083	.901	.201	1.135	.185
35	40	105	120	210	240	3.59	.220	.065	.093	.890	.206	1.139	.180
39	44	116	133	233	266	3.25	.219	.072	.104	.877	.209	1.137	.175
43	49	128	146	–	–	2.92	.218	.079	.116	.863	.213	1.136	.170
47	54	142	161	–	–	2.66	.218	.087	.130	.847	.216	1.130	.165
52	59	–	–	–	–	2.41	.218	.096	.144	.831	.219	1.125	.160
One-third.	–	$375 \div \frac{\rho_0 d.}{t_0}$.	–	Double.	–	Directly prop. to R_1, and subject to any multiplier less than given max.							–

TABLE B.

A Series of "TWO-NOSED CATENARIES" *inscribed in the Circle of Radius* (R_1) *Unity, and having Parallel Directrices at Graduated Distances* ($R_1 + Y_0$) *from its Centre from* (1 + .026) *to* (1 + .234).

This Table, which, when unfolded, fits on to each of the Supplementary Tables B_1, B_2, &c., has then for its purpose the designing of arch-rings, so as to secure the condition of the line of stress lying within the middle third, fifth, seventh, &c., of the arch-ring, as may be required to give strength *and* stability for every variation of proportion of parts and of the nature and distribution of load.

s.	θ_1.	θ_2.	m.	R_1.	R_2.	ρ_1.	$\rho_0 = \rho_2$.	y_0.	Y_0.	δ_0.	δ_2.
	° ′	° ′									
.230	21 – 29	32 – 13	.4890	1	1.0016	.9783	1.0201	.2345	.2340	.0007	.0003
.225	21 – 56	33 – 00	.4847	1	1.0017	.9761	1.0223	.2299	.2293	.0007	.0003
.220	22 – 24	33 – 48	.4805	1	1.0017	.9739	1.0245	.2254	.2247	.0008	.0004
.215	22 – 50	34 – 34	.4762	This circle forms the upper boundary of the "Kernel" of arch-ring, *i.e.* of its middle third, fifth, or seventh, &c.	1.0019	.9713	1.0271	.2208	.2200	.0009	.0004
.210	23 – 16	35 – 21	.4719		1.0020	.9687	1.0296	.2163	.2153	.0010	.0005
.205	23 – 41	36 – 06	.4675		1.0021	.9658	1.0327	.2117	.2106	.0011	.0005
.200	24 – 06	36 – 51	.4632		1.0022	.9628	1.0358	.2072	.2059	.0013	.0006
.195	24 – 30	37 – 35	.4588		1.0022	.9596	1.0391	.2026	.2011	.0015	.0007
.190	24 – 53	38 – 19	.4544		1.0023	.9563	1.0425	.1981	.1964	.0017	.0008
.185	25 – 15	39 – 02	.4499		1.0023	.9527	1.0463	.1935	.1916	.0019	.0009
.180	25 – 37	39 – 46	.4455		1.0024	.9491	1.0501	.1890	.1869	.0021	.0010
.175	25 – 58	40 – 28	.4410		1.0024	.9452	1.0543	.1845	.1821	.0023	.0011
.170	26 – 20	41 – 11	.4365		1.0024	.9413	1.0586	.1800	.1774	.0026	.0013
.165	26 – 40	41 – 53	.4319		1.0024	.9370	1.0635	.1754	.1726	.0028	.0015
.160	27 – 01	42 – 36	.4273		1.0024	.9327	1.0684	.1709	.1678	.0031	.0017
.155	27 – 20	43 – 18	.4226		1.0023	.9279	1.0738	.1664	.1629	.0034	.0019
.150	27 – 40	44 – 00	.4180		1.0022	.9232	1.0793	.1619	.1581	.0038	.0021
.145	27 – 59	44 – 42	.4132		1.0021	.9180	1.0856	.1574	.1532	.0042	.0023
.140	28 – 18	45 – 24	.4085		1.0020	.9129	1.0919	.1529	.1484	.0045	.0026
.135	28 – 36	46 – 06	.4036		1.0019	.9071	1.0990	.1483	.1434	.0049	.0029
.130	28 – 55	46 – 49	.3988		1.0017	.9014	1.1061	.1438	.1384	.0054	.0032
.125	29 – 12	47 – 31	.3938		1.0014	.8952	1.1142	.1392	.1333	.0059	.0035
.120	29 – 30	48 – 14	.3888		1.0011	.8890	1.1224	.1347	.1283	.0064	.0039
.115	29 – 47	48 – 57	.3836		1.0007	.8821	1.1318	.1301	.1231	.0070	.0043
.110	30 – 04	49 – 41	.3785		1.0004	.8752	1.1412	.1255	.1180	.0076	.0048
.105	30 – 20	50 – 25	.3731	The Radius of Described Circle is taken as unity.	0.9998	.8676	1.1521	.1209	.1127	.0082	.0053
.100	30 – 37	51 – 09	.3678		0.9993	.8600	1.1631	.1163	.1074	.0089	.0058
.095	30 – 52	51 – 54	.3622		0.9987	.8517	1.1760	.1116	.1019	.0097	.0064
.090	31 – 08	52 – 40	.3567		0.9981	.8433	1.1889	.1070	.0965	.0105	.0071
.085	31 – 23	53 – 27	.3508		0.9972	.8340	1.2043	.1023	.0908	.0114	.0079
.080	31 – 39	54 – 14	.3450		0.9963	.8246	1.2197	.0976	.0852	.0124	.0087
.075	31 – 54	55 – 03	.3388		0.9951	.8141	1.2384	.0928	.0793	.0134	.0097
.070	32 – 09	55 – 53	.3326		0.9940	.8036	1.2571	.0880	.0735	.0145	.0108
.065	32 – 23	56 – 45	.3259		0.9925	.7917	1.2803	.0831	.0673	.0158	.0121
.060	32 – 38	57 – 38	.3193		0.9910	.7798	1.3036	.0782	.0611	.0172	.0134
.055	32 – 52	58 – 34	.3121		0.9890	.7661	1.3335	.0732	.0544	.0188	.0150
.050	33 – 06	59 – 31	.3049	1	0.9870	.7524	1.3634	.0682	.0478	.0204	.0167
.045	33 – 20	60 – 34	.2968	1	0.9842	.7363	1.4037	.0630	.0406	.0223	.0190
.040	33 – 33	61 – 37	.2888	1	0.9815	.7203	1.4440	.0578	.0334	.0243	.0213
.035	33 – 46	62 – 49	.2808	1	0.9788	.7040	1.4843	.0526	.0260	.0263	.0236
Independent of R_1.			Directly proportioned to R_1, and subject to any multiplier.								

For the values of s ending with 5, on this and on the Supplementary Tables, the quantities are only interpolated as arithmetical means, and are correct to about 1 per cent.

IV.

THE BRAIN AND EYEBALL OF A HUMAN CYCLOPIAN MONSTER.

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Plates IV. and V.

[Read June 27, 1887.]

Some months ago a human cyclopian monster was presented by Dr. Mason, the master of the Coombe Hospital, to Dr. Bennett. Deformities of this nature cannot be said to be very uncommon, and the literature on the subject both in this country and abroad is most abundant. Still, as it is rare to get two examples precisely alike, and as in this specimen some very unusual characters were detected, we have thought that an account of the conditions presented by the brain and eyeball would not be devoid of interest.

The child, which was a female, had arrived at the full term of intra-uterine life. It presented no other deformities. According to Hannover, cyclopia occurs with much greater frequency in the female than in the male, and in this respect it agrees with anencephalus. Of the cases of cyclopia collated by this author, fifty-five were females and only sixteen males.*

Face.—A single ocular opening occupied the place which in a naturally-formed face corresponds to the root of the nose. This aperture was lozenge-shaped, and within it a large eyeball was apparent. But

* Mémoires de l'Académie Royale de Copenhague, 6me Série—Classe des Sciences, vol. i., No. 9.

although the opening was single, it was evident that the two palpebral fissures were engaged in its formation, seeing that it was surrounded by four eyelids. The nose, the place of which was usurped by the median eye, was absent as a projecting feature of the face. In cyclopians it is the rule that the displaced nose projects from the lower part of the forehead immediately above the eye, in the form of a pendulous and hollow snout, or proboscis. In place of this, however, there were, in this specimen, two puckered integumentary depressions, situated close together, on the lower part of the forehead, and above the palpebral opening. These constituted the only traces of a nasal organ which could be detected. The left nasal depression was somewhat deeper than the right.

The mouth, as is usual in such cases, was small, but the lunula of the upper lip, which, under ordinary circumstances, is developed from the central part of the embryonic fronto-nasal plate, was present. As we have not seen, either in Ahlfeld's great work * or elsewhere, any figure of a cyclopian which exactly corresponds with our specimen, we have reproduced a drawing of the head in Plate IV., fig. 1.

Brain.—As Forster, Cleland, and others have so well shown, the parts of the brain which are derived from the second and third primary cerebral vesicles are, in the cyclopian, for the most part fully represented, and quite natural. Not only this, but as a rule, as in this specimen, the nerves derived from these sub-divisions are present. These include all the cranial nerves from the third pair backwards. Still some failures in the development of the mesencephalic segment were noted. Thus, although the corpora quadrigemina were full and prominent—indeed relatively more so than usual—not a trace of the brachia, which in the normal brain extend forwards and upwards from them, could be detected. Further, the corpora geniculata interna were completely absent, and the crura cerebri were partially fused, and only slightly divergent.

It is in the parts which emanate from the anterior cerebral vesicle that the most striking alterations in the cyclopian brain are to be noted.

* Missbildungen des Menschen.

Thus the roof of the third ventricle is expanded into an enormous bladder-like dilatation filled with fluid. Cleland* gives an admirable illustration of this in his article upon the cyclopian brain, and he describes the condition as one of dropsy of the pineal or epiphysial diverticulum. But inasmuch as the pineal body was present in our specimen, and, further, inasmuch as this presented a distinct stalk, with a ventricular recess within it, we are inclined to consider the condition as one of dropsical expansion of the roof of the thalamencephalon in front of the epiphysis. Meckel† also noticed the presence of the pineal body in two cases; Eller‡ and Heuermann§ state that it was absent in the cases examined by them. The sacular dilatation was so extensive that it filled up the greater part of the cranial cavity, and stretched backwards over the cerebellum. Its thin epithelial walls were supported by the thick meninges of the brain, and joined the nervous lateral walls of the third ventricle (*i. e.* the optic thalami) along the strongly-developed pineal peduncles.

A prominent optic thalamus was present on each side (Pl. IV., fig. 3). Its anterior tubercle was fairly well marked, but on its hinder extremity there was not a trace of the pulvinar or of the corpus geniculatum externum. The outer surface was free, smooth, and bulging, whilst the crus cerebri supported its under surface. The enlarged pineal peduncle traversed its upper surface, and along this line the epithelial wall of the dropsical vesicle became continuous with the optic thalamus. Posteriorly the peduncles ran into the stalk of the pineal body, and concealed from view a feeble representative of the posterior commissure. Viewed from above, the two optic thalami appeared as if they were fused in the mesial plane—a shallow groove alone indicating the line of separation. They are described by Cleland as being "always joined into one," whilst Forster states that they are either fused or only separated by a shallow mesial furrow. This was not the case, however, in this specimen, because the apparent amalgamation was caused by an unusually strongly-

* *Journ., Anatomy and Physiology*, vol. xii.

† Ueber die Verschmeltzungsbildungen Meckel's Archiv. 1826–27.

‡ Mém. de Berlin, 1754. (Quoted by Meckel.)

§ Praktische Bemerkungen, Kopenhagen, Bd. i. (Quoted by Meckel.)

developed middle commissure, which stretched between the two optic thalami along their entire length, and served as a partition between the dropsical dilatation above, and a narrow ventricular fissure below. The aqueduct of Sylvius was widely dilated at its upper end, and its mouth was so placed with reference to the posterior margin of the middle commissure, that it opened partly into the dropsical cavity above, and partly into the inter-thalamic ventricular space below. It therefore acted as a means of communication between the two.

The parts derived from the floor of the thalamencephalic segment of the embryonic brain were for the most part present. Thus, there was a distinct and prominent tuber cinereum, but not a trace of an infundibulum, or of a pituitary body. According to Forster, the pituitary body is usually present. Two slightly marked, flattened elevations behind the tuber cinereum represented the corpora albicantia. In this case, however, the term applied to them is most inappropriate, because the white fibres of the anterior pillars of the fornix and bundles of Vicq d'Azyr were completely absent, and the prominences were evidently merely due to the grey nuclei of the corpora albicantia. The posterior perforated spot was also distinguishable, and presented the usual sieve-like appearance when the membranes and bloodvessels were withdrawn from it.

Only one optic nerve was present. It arose from the base of the brain in front of the tuber cinereum, or, in other words, from the place which in the normal brain is occupied by the optic chiasma. Not a trace of the optic tracts, nor indeed of any secondary connexion of the optic vesicle with the mesencephalon, could be detected. This is interesting when considered in connexion with the absence of the brachia of the corpora quadrigemina, the corpora geniculata, and the pulvinar—the parts with which the optic tract should stand in relation. This condition cannot be said to be universal in the cyclopian brain. In different cases very different parts of the optic vesicles are retained; in some cases it may be the fore parts fused together as in the present instance; in other cases the optic nerves may be absent, and only the optic tracts present. (Forster.)

Without doubt the cerebral mass constitutes the most remarkable part of the cyclopian brain. It is very small, its backward growth having been impeded by the dropsical expansion of the roof of the thalamencephalon. Further, the great longitudinal fissure and the falx cerebri are absent, and consequently the two hemispheres constitute one continuous mass with a single cavity—the representative of the two lateral ventricles. These are features which are constant in all cyclopian brains, and this specimen proved no exception to the general rule. Some characters, however, we noted which appear to be of considerable interest, and which, so far as we know, have not been hitherto observed.

The posterior surface of the cerebral mass was deeply hollowed out, and was closely applied to the anterior aspect of the dilated roof of the third ventricle. Along the upper margin of this surface the marginal convolution was clearly mapped out; and, owing to the absence of the great mesial fissure, it ran continuously across the middle line without break or interruption (Pl. IV., fig. 3, *h*). Below this, and separated from it by a shallow calloso-marginal sulcus, was the gyrus fornicatus (callosal convolution), which also passed continuously across the mesial plane (*a*). Posteriorly it turned downwards on each side, and then proceeded forwards upon the inner face of what apparently represented the temporo-sphenoidal lobe. Here it constituted a rudimentary hippocampal convolution. An elevated narrow ridge of grey matter extended from side to side below the callosal convolution, and ended laterally in a wider elevation immediately above the hippocampal convolution (*b*). This probably corresponds to the free edge of grey matter which, in the normal brain, lies at the bottom of the callosal fissure, whilst its wider extremity may be looked upon as representing the so-called fascia dentata. Faint traces of the calcarine, dentate, and collateral fissures were visible, as may be seen in figure 3 (Pl. IV.).

The corpus callosum, the anterior commissure, and the fornix were absent.

In front of the optic thalami, and below the ridge of grey matter, in connexion with the callosal convolution, a transversely oval aperture led

into the cavity of the combined lateral ventricles. The thin wall of the dropsical cavity was attached to the upper margin of this opening.

In shape the hemisphere-mass was conical. The broad base was directed backwards, and as we have already mentioned, it was hollowed out and fitted upon the front of the bladder-like dilatation of the third ventricle. The anterior extremity of the aborted cerebrum was prolonged forward in the form of two projections, partially separated from each other on the dorsal surface by a shallow mesial groove, which also proceeded backwards for a short distance on the upper surface of the mass. This fissure must be regarded as a faint indication of the mesial longitudinal fissure. The anterior projections which it separated were deflected slightly to the left, and ended in a knob-like mass of cerebral matter about the size of a small bean (Pl. IV., fig. 2, *i*). This was thrust forwards through a circular aperture in the lower part of the frontal plate of the frontal bone, and outside the cranial cavity it was placed in immediate relationship to the bottom of the left nasal integumentary depression. Indeed this little button-shaped protrusion of brain matter was merely separated from the subcutaneous tissue by the investing meninges.* Only one conclusion could be drawn from this curious condition, viz., that the arrangement represented a pair of fused and rudimentary olfactory lobes. In some respects they presented a resemblance to the olfactory bulbs of the fowl, if we could imagine these in a coalesced condition. But further evidence was afforded of the olfactory nature of these projections. The partially separated intra-cranial portions rested upon the orbital plates of the frontal bones, immediately over their line of union—the ethmoid bone being absent; and from the under surface of these a stream of slender meningeal processes passed downwards, and were connected to the dura mater covering the subjacent floor of the anterior cranial fossa. On raising the cerebrum these thread-like processes were very evident, and at first we regarded them as olfactory

* Since the above was written, Dr. Cunningham has seen in the pathological section of the Museum of the Royal College of Surgeons in England a specimen of a cyclopian pig, in which a rudimentary olfactory lobe was apparently thrust in a similar manner through the frontal bone into the root of the proboscis. (Teratological Series; No. 169 in the Catalogue.)

nerves, until the absence of the cribriform plate showed us that this was impossible, and an examination of their structure convinced us that they contained no nerve fibres. In all probability they represented the delicate coatings which the olfactory nerves, under ordinary circumstances, receive from the meninges, and it is indeed interesting to note the development of these tubular passages taking place independently of the nerves themselves.

The majority of authorities are agreed in regarding the absence of olfactory lobes and nerves as a characteristic of the cyclopian brain. Hannover, in the elaborate memoir already referred to, remarks: "Les bulbes olfactifs, le corps calleux et la voûte manquent." Cleland * states that in cyclopia there is not a vestige of the olfactory bulbs to be seen, and Forster † would seem to infer the same thing. On the other hand, Jourdan ‡ mentions a case in which "olfactory nerves" were present. We have not had an opportunity of consulting the Paper of this author, but in Ahlfeld's work § it is referred to, and five of Jourdan's figures are reproduced in plate XLVI. Thus, fig. 8, in this plate, represents the base of the brain, and fig. 9 the floor of the cranium, and in both of these olfactory nerves are somewhat obscurely depicted. All this serves to show how very varied the characters are which may be presented by different cases of cyclopia. At the same time it is evident that olfactory lobes, even in the most rudimentary form, must occur very seldom, because Hannover's results are based upon the systematic examination of no fewer than eighteen specimens.

On the outer surface of the hemisphere-mass of this specimen a rounded and prominent island of Reil could be distinguished (Pl. IV., fig. 2, *a*). It was completely exposed, and the upper and lower lips of the Sylvian fissure, which were only faintly indicated, were, in consequence,

* "Contribution to the Study of Spina Bifida, Encephalocele, and Anencephalus," *Journ. Anat. and Phys.*, vol. xvii.

† Die Missbildungen des Menschen, Jena, 1885.

‡ Description Anatomique d'un cas de Cyclopie. (Quoted by Ahlfeld.)

§ Die Missbildungen des Menschen, 1882. Pl. XLVI.

widely separated from each other. So far as we could discover, the corpus striatum was absent.

The single ventricular cavity in the hemisphere-mass turned outwards and backwards on each side, so as to tunnel the entire length of each hemisphere. Its walls were smooth, with no elevations which could be referred to the hippocampal or other eminences present in the normal brain.

Eyelids and Lachrymal Apparatus.—As we have previously stated, the large single palpebral opening was bounded by the four eyelids. These were united in such a manner as to form a lozenge-shaped opening, produced by the coalescence of the two palpebral fissures at their inner extremities. The lateral angles of the opening corresponded therefore with the outer canthi; and extending outwards from each of these, as is usual in such cases, there was a deep integumentary furrow (Pl. IV., fig. 1). The combined carunculæ lachrymales and plicæ semilunares were retained in connexion with the lower angle of the palpebral opening, but were completely absent at the upper angle. The arrangement which we have described may be regarded as the more usual condition of the eyelids in specimens of cyclopia, which are furnished with a distinct eyeball. Many variations, however, are met with, and cases are recorded in which the eyelids have been completely absent.*

Two small lachrymal glands were observed in the fore-part of the orbital cavity—one upon either side of the single eye-ball. As a general rule, these glands are present in cyclopian monsters; because, as Meckel remarks, "durch die Verschmeltzung der innern Winkel, die äussere Gegend des Auges nicht afficirt ist."

On the inner extremity of each lower eyelid a distinct punctum lachrymale was visible. On the upper eyelids they were absent. Each punctum led into a canaliculus, and this in turn opened into a small lachrymal sac. The lachrymal sacs were two in number, and were lodged in two depres-

* Heuermann: Practische Bemerkungen, Kopenhagen. Bd. i.

* Hooper: *Memoirs of the London Medical Society*, vol. ii.

sions on the front part of the floor of the common orbit—one on each side of the mesial plane. A rudimentary lachrymal bone entered into the formation of each depression. The right sac ended blindly, whilst the left was prolonged in the form of a narrow channel downwards and backwards through the floor of the orbit, and finally opened upon the roof of the mouth. Meckel* describes a cyclopian lamb in which the puncta lachrymalia were present on the upper eyelids, but absent on the lower lids; and he quotes from Sonsis, who observed them in one instance upon both the upper and lower eyelids.

Orbital Muscles.—On removing the roof of the single orbital cavity, two superior oblique muscles came into view. They lay side by side, and their identity was placed beyond a doubt from the fact that the fourth nerve was traced into each, and also from the presence of two pulleys through which the tendons of the muscles passed. These tendons crossed each other on the upper aspect of the eyeball. The right tendon was inserted into the left side of the eyeball and the left tendon into the right side of the eyeball. Not a trace of the levatores palpebrarum nor of the superior recti could be detected. On each side of the globe of the eye there was a large and powerful external rectus muscle, which was entered by the sixth nerve.

On tilting forward the optic nerve and eyeball a complicated muscular arrangement was displayed (Pl. IV., fig. 4). Two muscles, which resembled the external recti, were placed in the angles formed by the floor and outer walls of the orbit (*b*). These were supplied by branches from the third nerve (*a*), and therefore we conclude that they were the representatives of the inferior recti. Between these muscles, and lying upon the floor of the orbit, there was a curious cruciform muscle. The transverse band of this was fleshy and strong, and passed under the eyeball from one inferior oblique muscle to the other, and to the ocular surfaces of these its fibres were attached (*d*). It supported the globe of the eye like a sling, but had no direct attachment to it. From the back o the orbit a slender tendinous band came forward in the mesial

* Ueber die Verschmeltzungsbildungen, Meckel's Archiv., 1826–27, p. 246.

plane, and joined the posterior margin of the transverse muscle; whilst from the anterior margin of the transverse band a delicate fleshy fasciculus ran forward also in the mesial plane, and ended in the connective tissue surrounding the two lachrymal sacs. The third nerves were closely associated with this peculiar muscular apparatus. The two nerves entered the orbit close to each other, and, when fairly within, they were joined across the middle line by a short stout transverse branch, or commissure.* Immediately beyond this each nerve gave a branch to the corresponding inferior rectus, and then proceeded onwards. Close to the transverse muscular band they broke up into numerous large branches which entered this muscle. One slender twig from each was continued forwards towards the orbital opening, but the distribution of this was not made out. We believe that it supplied the anterior mesial muscular fasciculus.

The question now is, To which of the normal orbital muscles can we refer the different parts of the cruciform muscular arrangement on the floor of the cyclopian orbit? From the greater part of both the third nerves entering the transverse band we consider that in all probability it represented the fused inferior oblique muscles. The mesial slips attached to this may be regarded as the aborted and coalesced internal recti muscles.

Meckel describes the muscles of the orbital cavity in a cyclopian pig with an eyeball formed somewhat in the same manner as that of this specimen. He states that all the muscles were present, with the exception of the internal recti; but he does not mention the arrangement of the nerves of supply. Further, from the description which he gives of the inferior rectus, we infer that the muscular band to which he gives this name is that which we have referred to the inferior oblique. He describes it as "aus Querfasern gebildeter, nirgends an den Knochen gehefteter."

Eyeball and Optic Nerve.—The optic nerve was single throughout its whole length. It took origin from the base of the brain, in front of the

* This commissural arrangement in the third nerves was steeped in glacial acetic acid, and teased with needles. When placed under the microscope the most complete degree of intercrossing of fibres between the two nerves was seen to take place in the commissure.

tuber cinereum. In was uniform in girth until it reached a point a short distance behind the globe of the eye, where it suddenly expanded previous to its entrance into the eyeball (Pl. IV., fig. 5).

A series of transverse sections were made through the optic nerve at different points from its origin to its termination (Pl. V., figs. 1 to 4). On examining these sections under the microscope the connective tissue framework of the nerve is seen to be very coarse. The septa are thick, and the compartments which they form larger than in the normal optic nerve. Altogether, the connective tissue elements of the nerve are much increased in quantity. Again, in the centre of the nerve there is a collection of connective tissue forming a core. This is present throughout the whole length of the nerve, and appears to be independent of the connective tissue carried in from the pia mater by the central vessels. The central artery and vein are single, and after their entrance the connective tissue core is considerably increased (Pl. V., fig. 3).

The nerve fibres appear to be normal; but, owing to the increase in the quantity of the connective tissue, they are not so numerous as in an ordinary optic nerve. In figs. 1, 2, and 3., Pl. V., they are only indicated over a small area of each section, in order that in the remainder of the drawing the characters of the connective tissue framework may be the more readily appreciated.

In the anterior expanded portion of the optic nerve the connective tissue core increased greatly, and formed at least one-third of the entire bulk of the nerve. The central vessels were very large, and an extensive fissure was noticed between the artery and vein. This could be detected more or less distinctly in the entire series of sections through this portion of the nerve. Its presence is difficult to explain. It is possible that it may indicate a partial non-coalescence of the two primitive vesicles out of which the nerve is formed. We know that in cyclopia the optic nerve has occasionally been observed to divide, after it has entered the eyeball, into two portions, each of which forms a separate retina. Again, it is not unusual for an optic nerve, which is single behind, to divide in front, and to make its entrance at two distinct points. (Meckel.)

As we shall afterwards see, the eyeball of our specimen was provided

with two retinal membranes. The right retina was much more extensive than the left, and it was to this that the large central artery was ultimately traced.

Fig. 4, Pl. v., gives a view of the connective tissue core and central blood-vessels, as seen in one of the sections through the expanded part of the optic nerve close to the eyeball. The magnifying power employed was about × 41.

In cyclopian monsters the highest degree of variation is found in the eyeball. Huschke* recognises no fewer than six main varieties with many intermediate gradations. It will simplify our further description if we briefly enumerate these:—(1) The first grade is where the eyeball is completely absent. This is not a very common variety, but still numerous cases have been recorded, the most recent being that described by Symington and Woodhead.† (2) One eyeball in a single orbit, all the parts of the visual apparatus being single. This form is more frequent than the foregoing. (3) The two eyeballs more or less fused, some of their parts being single and some double. This is the most common form. (4) The two eyeballs connected, but all the parts of each present. (5) Two separate eyeballs in a common orbital cavity. (6) Two eyeballs, each in its own orbit.

The eyeball of our specimen belonged to Huschke's third grade. It was very large, and transversely oval in shape (Pl. IV., fig. 5). The sclerotic formed a continuous coat around the entire globe, but in front the cornea was double—that on the right side being slightly larger than that on the left side. On opening up the eyeball, a salient ridge upon the upper and anterior aspects of the deep surface of the sclerotic indicated the line of junction of the two globes. The choroid was single, but behind each cornea there was a distinct and well-formed iris and pupil (fig. 6). Two ciliary bodies were also present. The retina was double, and therefore formed two chambers within the eyeball. It was not possible by dissection to separate the two retinae from each other. One layer

* Ueber die erste Entwickelung des Auges und die damit zusammenhängende Cyclopie.—Meckel's Archiv., 1830-32.

† "Observations on Cyclopia."—*Proc. Roy. Phys. Soc. Edinb.*, 1886.

only occupied the mesial plane, and from this the two laminae which enclosed the ocular chambers appeared to spring (fig. 6). The projecting ridges of sclerotic already referred to were received between the two retinal layers as they diverged from the mesial septum. In each chamber there was a crystalline lens and vitreous body. The left chamber was distinctly smaller than the right. This was all the information which we obtained by dissection, but a few additional facts were acquired by means of the microscope.

A portion of the upper part of the eyeball having been removed (Pl. IV., fig. 5), a series of sections were made through the tunics. The retina was then seen to possess all its layers. The elements composing the nerve-fibre and ganglionic strata seemed to be somewhat deficient in quantity, but the other layers were perfectly normal. The sustentacular fibres of Müller were particularly distinct, and their expanded bases stood out in a very prominent manner. Altogether they appeared more plentiful than in the normal retina; but this appearance may have been caused by the deficiency in the two deeper retinal strata.

A portion of the retinal septum was then removed and subjected to the same treatment. The appearances produced by these sections were somewhat puzzling. That this septum was composed of retinal elements there could not be the least doubt, but these were arranged in a very confused manner, and there was little trace of the normal stratified disposition of parts (Pl. V., fig. 5, *a*). Further there were few features in the sections which tended to demonstrate that the partition was formed by the fusion of the two retinal membranes in the mesial plane. Bloodvessels, it is true, were present upon both surfaces, but on the other hand it was only on the right side that the expanded feet of the Müllerian fibres could be detected. Not a trace of the pigmentary layers nor of the rods and cones was visible.

A continuous series of sections were lastly made in a horizontal direction through the fundus of the eyeball at the optic nerve entrance.

In these sections some interesting points were made out. Stretching outwards from each side of what in the normal eye corresponds with the

optic disc, the lateral portions of retina which enclose the separate ocular chambers could be seen (Pl. v., fig. 5, *c* and *k*; fig. 6, *b* and *o*); whilst extending forwards from the centre of the optic entrance was the mesial retinal septum (figs. 5 and 6, *a*). When the highest sections were examined (viz. those which passed through the fibrous sheath of the nerve, but did not involve the optic nerve fibres), the right lateral retina was seen to be in more or less direct continuity with the mesial retinal septum by some scattered clusters of retinal granules which extended across the interval between them (fig. 5). The left retina, on the other hand, was completely cut off from the mesial retina, and showed no continuity with it. Fig. 5, Plate v., is a drawing taken from one of these higher sections under a magnifying power of × 12½. The fibrous coat of the optic nerve is seen blending with the sclerotic. The clusters of granules between the right lateral retina and the mesial retinal septum are not so well marked as in sections a little higher. The section figured corresponds to the upper margin of the optic disc.

Another point of some interest may be observed in this series of horizontal sections through the fundus of the eyeball. A slender strand of connective tissue is continued forward upon the left side of the retinal partition (fig. 5, *b*; fig. 6, *p*). This can be traced through the entire series of the sections, and must therefore be regarded as a fibrous septum supplementing the retinal partition. In figure 5, Pl. v., it is seen to be continuous behind with the fibrous tissue of the sclerotic. Lower down it became much finer, but at the centre of the optic entrance it again enlarged, and was then seen to be continuous with the entral core of connective tissue in the interior of the optic nerve (Pl. v., fig. 6, *p*).

The presence of a fibrous septum in this position would seem to imply that the retinal partition was formed by the right retina alone, and that the left retina was absent in the mesial plane. The following facts appear to favour this view:—(1) At the fundus of the eyeball, the left retina is separated from the retinal partition by the broad base of the fibrous partition; (2) the right lateral retina has no such barrier interposed

between it and the retinal septum, but is more or less directly continuous with it; (3) in the septal retina, microscopical examination showed that the broad feet of Müller's fibres were only present on the right surface of the partition.

On the other hand, we have the evidence afforded by dissection, which seemed to indicate that the septal retina split into the two enclosing membranes in the upper, lower, and front parts of the eyeball. It is possible, however, that had we been able to subject this apparent splitting of the retinal partition to the microscopical test, we might have failed to establish the continuity of the different parts of the membrane.

But we have still to account for the presence of the fibrous septum which was placed on the left side of the retinal partition, and which posteriorly was directly continuous with the sclerotic and central core of connective tissue in the optic nerve. At first we were inclined to regard it as a vestige of the inner parts of the two sclerotic tunics fused together. Its position to the left of the right retina, and in the interval between the two vitreous bodies, seemed to favour this view. But Professor Cleland, who was so good as to examine our microscopical sections, suggested another explanation. He consider that it stands in connexion with the central artery of the retina, and is therefore in some measure analogous to the septum of fibrous tissue which is led into the interior of the optic nerve of the ox.

We have already stated that the central artery, which was so evident in the sections through the optic nerve, was traced to the right retina. But in some of the sections through the fibrous septum in question a small vessel, cut in its long axis, was distinctly visible. This may have been the representative of the central artery of the left retina.

Another point was brought out by the horizontal sections through the fundus of the eyeball, viz., that a much greater number of optic nerve-fibres entered the right retina than the left retina (Pl. v., fig. 6). Broadly speaking, the proportion might be said to be about two-thirds for the

right retina and one-third for the left retina. This of course was due to the smaller size of the left retina and its absence in the mesial plane.

Figure 6, Pl. v., gives a view of a horizontal section through the fundus of the eyeball at the level of the central point of the optic nerve entrance. The bundles of optic nerve-fibres are observed to be cut transversely. The fibrous partition on the left of the mesial retinal septum may be noticed, and also the manner in which it is continuous with the central core of fibrous tissue in the optic nerve.

In the series of sections about this level the two ciliary ganglia were divided. They lay closely applied to the sclerotic coat of the eyeball—one on each side of the optic nerve. The left ganglion was much smaller than the right—probably in correspondence with the smaller dimensions of the left portion of the eyeball (Pl. v., fig. 6, *g* and *k*). In structure these ganglia were normal.

The causes at work in producing the remarkable series of deviations from the normal condition in cyclopian monsters have engaged the attention of numerous observers, and foremost amongst these we may mention the names of Cleland and Hannover. It would be rash for us, with our limited experience of one case, in which the examination has been confined entirely to the brain and visual apparatus, to discuss this problem from the stand-point of our own observations.

Professor Cleland considers that the defect in the brain and the deformity of the face probably depend upon a common cause: viz., the constriction exercised by a diseased vessel situated in the groove between the first and second cerebral vesicles.* A dropsical condition of the roof of the thalamencephalon results, and at the same time "a downward curvature, as well as a rotation inwards, of the lateral parts for the growth

* *Journal of Anatomy and Physiology*, vol. xii., p. 523.

of the face" is induced.* The primary optic vesicles are consequently pressed together and ultimately included within a common globe, whilst the fronto-nasal plate is thrust forwards above the mesial eye as a proboscis.

Hannover, who has contributed the most recent memoir upon the subject, takes a different view of the case. He believes that certain of the conditions present in cyclopian monsters are due to an arrest in development, while others are produced by an arrest in formation ("arrêt de formation"). The defects in the brain he considers in no respect dependent upon the coalescence of the two eyes, but to arise from an arrest in development. On the other hand, seeing that the eye is primarily double, its deformity is the result of a failure of formative material in those parts of the eyes "which are turned towards each other." The defect in the skull can be referred to the same cause. The achordal portion of the primordial basal cartilage is absent or incomplete through a failure of the materials required for its formation. Referring to the abnormal position of the nose, he says: "La véritable raison pour laquelle le nez deforme des fœtus cyclopiens avec un œil unique est situé au-dessus de l'œil doit être cherchée dans le cartilage primordial, sur lequel le cerveau repose directement. Dans les conditions normales, la première vésicule cérébrale s'avance entre les vésicules des yeux, et le cartilage primordial doit nécessairement l'accompagner puisqu'il constitue la base de la vésicule cérébrale; mais lorsqu'il s'est formé un œil unique dans la ligne médiane du corps et que la place est pour ainsi dire prise, le cartilage primordial est forcé d'en prendre une qui sera toujours au-dessus de l'œil." †

Hannover has drawn attention to the fact that in cyclopian monsters a round cæcal pit may be noticed at the bottom of the pituitary fossa. In thirty normal fœtuses he only observed this canal present in one specimen. It is a vestige of the passage through which the alimentary part of the pituitary body and the buccal cavity were originally connected.

* *Journal of Anatomy and Physiology*, vol. xvii.

† Mémoires de l'Acad. Roy. de Copenhague, 6me série—Classe des Sciences. Vol. i., No. 9.

Hannover, therefore, gives it the name of Rathke's canal. It was present in a well-marked form in this specimen.

In conclusion, we must express our thanks to our colleague, Professor Purser, for the assistance he gave us in the microscopical examination of the eyeball and optic nerve. The figures in Plate v. were drawn for us by Dr. H. C. Earl, Demonstrator of Physiology in Trinity College, and we are greatly indebted to him for the careful and accurate manner in which he has delineated the various objects.

DESCRIPTION OF PLATE IV.

PLATE IV.

FIGURE 1.—The face of the Cyclopian Monster described in the text. In place of the usual proboscis there are two nasal integumentary depressions in the lower part of the forehead immediately above the palpebral opening. The olfactory lobe of the brain was thrust through an aperture in the frontal bone, and came into relationship with the left of these depressions.

FIGURE 2.—The base of the brain. The meninges are represented as being still in place in the interpeduncular space.

a. Island of Reil.
b. Temporo-sphenoidal lobe.
c. Fifth cranial nerve.
d. Sixth nerve.
e. Auditory and facial nerves.
f. Fourth nerve.
g. Third nerve.
h. Single optic nerve surrounded by its membranes.
i. Olfactory lobe.

FIGURE 3.—View of the brain from behind and above after the removal of the dropsical dilatation of the roof of the third ventricle.

a. Gyrus fornicatus.
b. Fascia dentata.
c. Calcarine fissure.
d. Dentate fissure.
e. Cerebellum.
f. Corpora quadrigemina.
g. Pineal peduncle.
h. Marginal convolution.

FIGURE 4.—Cruciform muscular arrangement on the floor of the orbit.

a. Third or oculo-motor nerve.
b. Inferior rectus.
c. Lower eyelids.
d. The fused inferior oblique muscles?

FIGURE 5.—The eyeball opened from above. The two retinal chambers may be seen.

FIGURE 6.—The upper and front part of the globe of the eye has been removed, and the lens and vitreous body taken out of each retinal chamber. The iris and pupil are seen, also the mesial partition formed by the right retina.

Fig. 1.

Fig. 2.

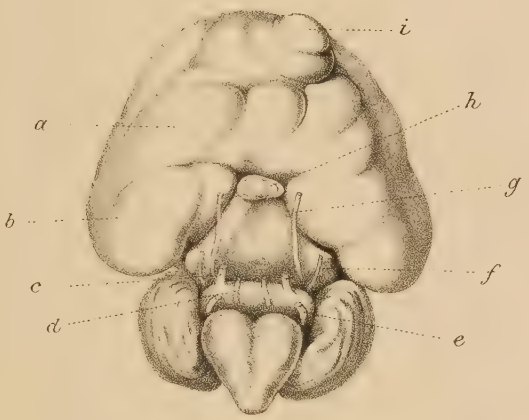


Fig. 3.

Fig. 4.

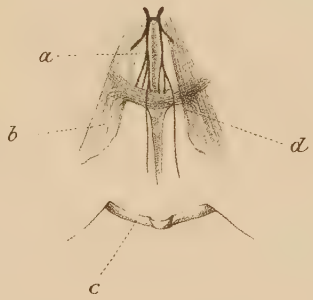


Fig. 5

Fig. 6.

F. Huth, Lith.r Edin.r

DESCRIPTION OF PLATE V.

PLATE V.

[This Plate is the work of Dr. H. C. Earl, Demonstrator of Physiology, Trinity College.]

FIGURES 1, 2, 3.—Transverse sections through the optic nerve at different points. Fig. 1 is a section through the nerve, close to its origin from the base of the brain; fig. 2, a section at a point midway between its origin and its termination in the two retinæ; fig. 3, a section through the nerve, close to the point where it expands before entering the eyeball. In each of these the nerve-fibres are only represented over a small area of the section, so that the trabecular arrangement of the connective tissue may be the more readily appreciated.

FIGURE 4.—The central connective tissue core of the anterior expanded part of the optic nerve as observed under a magnifying power of × 41. The central vessels and a space between them are seen.

FIGURE 5.—Horizontal section through the fundus of the eyeball at the level of the upper margin of the optic disc. The sheath given to the optic nerve by the dura mater is divided, but none of the optic nerve-fibres.

a. Retinal partition.
b. Fibrous partition: note its continuity with the sclerotic.
c. Left lateral retinal membrane.
d. Pigmentary layer.
e. Choroidal bloodvessels.
f. Sclerotic coat.
g. Cluster of retinal granules in the interval between the retinal partition and the right retinal membrane. In the higher sections these clusters were more numerous, and brought the septal and right lateral portions of the retina into continuity with each other.
h. Choroidal bloodvessels.
i. Pigmentary layer.
k. Right lateral retinal membrane.

FIGURE 6.—Horizontal section through the fundus of the eyeball, near the centre of the optic disc. The unequal distribution of the optic fibres to the two retinæ is seen, and also the manner in which the larger right bundle of nerve-fibres is in connexion both with the retinal septum and the right lateral retinal membrane. Note that the central core of connective tissue is in continuity with the fibrous partition which supports the retinal septum on the left.

a. Retinal septum.
b. Left lateral retinal membrane.
c. Wrinkle or fold of the retina as seen in section.
d. Pigmentary layer.
e. Choroidal bloodvessels.
f. Sclerotic coat.
g. Left lenticular ganglion.
h. Central connective tissue core of optic nerve.
i. Optic nerve-fibres at optic disc.
k. Right lenticular ganglion.
l. Sclerotic coat.
m. Choroidal bloodvessels.
n. Pigmentary layer.
o. Right lateral retinal membrane.

Fig. 1.
(x 12½)

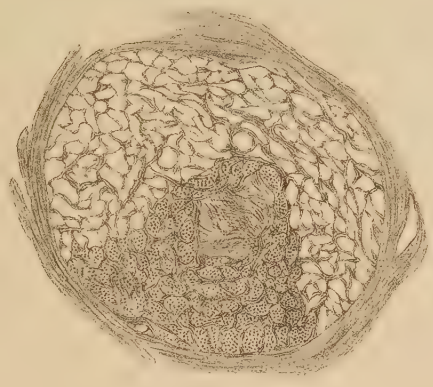


Fig. 5.
(x 12½)

Fig. 3.
(x 12½)

Fig. 6.
(x 12½)

Fig. 4.
(x 41)

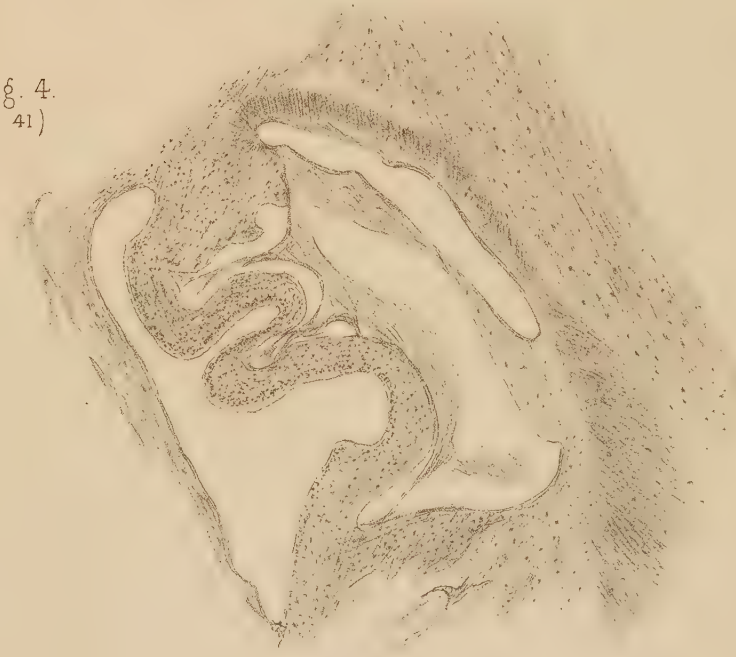
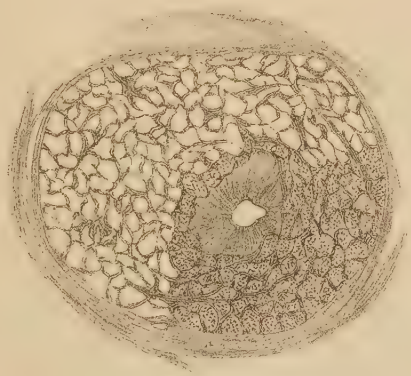


Fig. 2.
(x 12½)



F. Huth, Lith^r Edin^r

V.

ON THE THEORY OF THE CONTENT. By SIR ROBERT STAWELL BALL., LL.D., F.R.S., Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.

[Read December 12, 1887.]

CONTENTS.

		PAGE
Introduction,		123
I.	Preliminary Notions,	124
II.	On the Intervene between Two Objects in a Range,	126
III.	On the Intervene between Two Objects in an Extent,	135
IV.	On the Intervene between any Two Objects in a Content,	137
V.	On the Theory of Departure,	143
VI.	On the Representation of the Objects in a Content by the Points in Space,	149
VII.	On the Homographic Transformation of the Content,	151
VIII.	Application of the Orthogonal Transformation,	164
IX.	The Theory of the Vector,	171
Conclusion,		182

Introduction.

In the study of the so-called "Non-Euclidian Geometry" I have often felt a difficulty which has, I know, been shared by others. In that Theory it seems as if we try to replace our ordinary notion of distance between two points by the logarithm of a certain anharmonic ratio. But this ratio itself involves the notion of distance measured in the ordinary way. How, then, can we supersede our old notion of distance by the non-Euclidian notion, inasmuch as the very definition of the latter involves the former?

In spite of this difficulty it seemed impossible to doubt the instructive character of the non-Euclidian conceptions, and the interest of the extensive theory to which they give rise.

The object of the earlier part of the present Paper is to show the way in which, in my own meditations, I have got rid of the difficulty. I was partly guided by the Ausdehnungslehre to form a conception of what, for the want of a better term, I have called the *content.* I understand by this word what Grassman would call the Gebiet of four Extensiver Grössen, and the several Grössen forming the Gebiet I call the *objects* of the content. I now correlate the several objects of the content with the points of space. The conceptions of the so-called non-Euclidian Geometry are just fitted for the study of the content, while by a system of correspondence the calculations are conducted in common space, and the confusion already spoken of disappears.

I should add that the Ausdehnungslehre has been already applied to the subject by Mr. Homersham Cox and by Mr. Buchheim,* but their writings have involved far more of Grassman's theory than is required for the simple initial conception that I here derive from it. I must also refer to Professor Heath's beautiful discovery in the Theory, while I am, of course, largely indebted to the classical writings of Gauss, Cayley, Riemann, Clifford, Klein, and Lindemann.

I believe, however, that the general treatment of the subject as here given is novel, and that the Theory of the Vector contains some theorems which appear to me to be important, and which I have not seen published before.

I.—PRELIMINARY NOTIONS.

§ 1. **Definitions.**—Let a_1 be a certain symbol denoting an *object;* let a_2, a_3, a_4 be other symbols denoting other objects. Let x_1, x_2, x_3, x_4 be four numerical magnitudes of any description. Then the symbolical expression,

$$x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4,$$

denotes what we may call the *object* X.

In a similar way the object Y will be expressed by

$$y_1a_1 + y_2a_2 + y_3a_3 + y_4a_4.$$

* Since these words were written, death has removed Mr. Buchheim from us at a very early age. We have to deplore the loss of a consummate mathematician and a profound scholar.

The object X and the object Y will be identical if, and only if, the equations

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \frac{x_4}{y_4}$$

are fulfilled.

All the different objects obtained by varying the ratios of x_1, x_2, x_3, x_4 in the expression

$$x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4,$$

form what we call the 'content.'

Any object in the content may be conveniently specified by its four co-ordinates, x_1, x_2, x_3, x_4, but the object defined by λx_1, λx_2, λx_3, λx_4 is the same, whatever be the value of λ.

All the objects denoted by

$$x_1a_1 + x_2a_2,$$

as the ratio $x_2 \div x_1$ varies, form what is called a *range.*

All the objects denoted by

$$x_1a_1 + x_2a_2 + x_3a_3,$$

as the ratios $x_2 \div x_1$ and $x_3 \div x_1$ vary, form what is called an *extent.*

§ 2. **Theorems.**—All the objects in an extent of which the co-ordinates x_1, x_2, x_3 satisfy a linear equation lie on a range, for then $x_3 = p_1x_1 + p_2x_2$, and the symbol

$$x_1a_1 + x_2a_2 + x_3a_3$$

becomes

$$x_1(a_1 + p_1a_3) + x_2(a_2 + p_2a_3);$$

and as $x_2 \div x_1$ varies, this forms a range defined by the two objects,

$$a_1 + p_1a_3 \quad \text{and} \quad a_2 + p_2a_2,$$

which, of course, belong to the extent, inasmuch as the first is obtained by making $x_1 = 1$, $x_2 = p_1$, $x_3 = 0$, and similarly for the second.

We can prove, in like manner, that all the objects in a content, of which the four co-ordinates satisfy a linear equation, lie in the same extent.

It also follows that the objects in a content whose co-ordinates satisfy two linear equations lie on a range.

If the co-ordinates of an object in an extent satisfy a homogeneous equation of the nth degree, we say that they form a range of the nth degree. If the co-ordinates of an object in a content satisfy an equation of the nth degree, then those objects constitute an extent of the nth degree.

It must be noticed that the content, with its objects, ranges, and extents, have no necessary connexion with *space.* It is only for the sake of studying the content with facility that we correlate its several objects with the points of space.

II.—ON THE INTERVENE BETWEEN TWO OBJECTS IN A RANGE.

§ 3. **The Intervene.**—In ordinary space the most important function of the co-ordinates of a pair of points is that which expresses their distance apart. We desire to create that function of a pair of objects which shall be homologous with the distance function of a pair of points in ordinary space.

The nature of this function is to be determined solely by the attributes which we desire it to possess. We shall take the most fundamental properties of distance in ordinary space. We shall then re-enunciate these properties in generalized language, and show how they suffice to determine a particular function of a pair of objects. This we shall call the *Intervene between the Two Objects.*

Let P, Q, R be three collinear points in ordinary space, Q lying between the other two; then we have, of course, as a primary notion of distance,

$$PQ + QR = PR.$$

In general, the distance between two points is not zero, unless the points are coincident. An exception arises when the straight line joining the points passes through either of the two circular points at infinity. In this case, however, the distance between every pair of points on the straight line is zero. These statements involve the second property of distance.

In ordinary geometry we find on every straight line one point which is at an infinite distance from every other point on the line. We call this the point at infinity. Sound geometry teaches us that this single point is properly to be regarded as a pair of points brought into coincidence by the assumptions made in Euclid's doctrine of parallelism. The existence of a

pair of infinite points on a straight line is the third property which, by suitable generalization, will determine an important feature in the range. The fourth property of ordinary space is that which asserts that a point at infinity on a straight line is also at infinity on every other straight line passing through it. This obvious property is equivalent to a significant law of intervene which is vital in the theory. If we might venture to enunciate it in an epigrammatic fashion, we would say that there is no short cut to infinity.

The fifth property of common space which we desire to generalize is one which is especially obscured by the conventional coincidence of the two points at infinity on every straight line. We prefer, therefore, to adduce the analogous, but more perfect, theorem relative to two plane pencils of homographic rays in ordinary space, which is thus stated. If the two rays to the circular points at infinity in one pencil have as their correspondents the two rays to the circular points in the other pencil, then it is easily shown that the angle between any two rays equals that between their two correspondents.

We now write the five correlative properties which the intervene is to possess. They may be regarded as the axioms in the Theory of the Content. Other axioms will be added subsequently.

§ 4. **First Group of Axioms of the Content.—**

I. If three objects, P, Q, R on a range be ordered in ascending parameter, then the intervenes PQ, QR, PR are to be so determined that

$$PQ + QR = PR.$$

II. The intervene between two objects cannot be zero unless the objects are coincident, or unless the intervene between every pair of objects on the same range is also zero.

III. Of the objects on a range, two either distinct or coincident are at infinity, *i. e.* have each an infinite intervene with all the remainder.

IV. An infinite object on any range has an infinite intervene from every object of the content.

v. If the several objects on one range correspond one-to-one with the several objects on another, and if the two objects at infinity on one range have as their correspondents the two objects at infinity on the other, then the intervene between any two objects on the one range is equal to that between their correspondents on the other.

§ 5. **Determination of the Function expressing the Intervene between any Two Objects on a Given Range.**—Let x_1, x_2, x_3, x_4, and y_1, y_2, y_3, y_4 be the co-ordinates of the objects by which the range is determined. Then each remaining object is constituted by giving an appropriate value to ρ in the system,

$$x_1 + \rho y_1, \quad x_2 + \rho y_2, \quad x_3 + \rho y_3, \quad x_4 + \rho y_4.$$

Let λ and μ be the two values of ρ which produce the pair of objects of which the intervene is required. It is plain that the intervene, whatever it be, must be a function of x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4, and also of λ and μ. As far as objects on the same range are concerned, we may treat the co-ordinates of the originating objects as constant, and regard the intervene simply as a function of λ and μ, which we shall denote by $f(\lambda, \mu)$.

The form of this function will be gradually evolved, as we endow it with the attributes we desire it to possess. The first step will be to take a third object on the same range for which the parameter shall be ν, where λ, μ, ν are arranged in order of magnitude. Then, as we wish the intervene to possess the property specified in Axiom I., we have

$$f(\lambda, \mu) + f(\mu, \nu) = f(\lambda, \nu).$$

By the absence of μ from the right-hand side, we conclude that μ must disappear identically from the left-hand side. This must be the case whatever λ and ν may be. Hence, no term in which μ enters can have λ as a factor. It follows that $f(\lambda, \mu)$ must be simply the difference of two parts, one being a function of λ, and the other the same function of μ.

Accordingly, we write,

$$f(\lambda, \mu) = \phi(\lambda) - \phi(\mu).$$

The first step in the determination of the intervene function has thus been taken. But the form of ϕ is still quite arbitrary.

The rank of the objects in a range may be concisely defined by the magnitudes of their corresponding values of ρ. *Three sets are said to be ordered when the corresponding values of ρ are arranged in ascending or descending magnitude.*

Let P, Q, Q' be three ordered objects, then it is generally impossible that the intervenes PQ and PQ' shall be equal; for, suppose them to be so, then

$$PQ + QQ' = PQ' \quad \text{by Axiom I.;}$$

but, by hypothesis, $$PQ = PQ',$$

and hence $$QQ' = O.$$

But, from Axiom II., it follows that (Q and Q' being different) this cannot be true, unless in the very peculiar case in which the intervene between every pair of objects on the range is zero. Omitting this exception, to which we shall subsequently return, we see that PQ and PQ' cannot be equal so long as Q and Q' are distinct.

We hence draw the important conclusion that there is for each object P but a single object Q, which is at a stated intervene therefrom.

Fixing our attention on some definite value δ (*what* value it does not matter) of the intervene, we can, from each object λ, have an ordered equi-intervene object μ determined. Each λ will define one μ. Each μ will correspond to one λ. The values of λ with the correlated values of μ form two homographic systems. The relation between λ and μ depends, of course, upon the specific value of δ, but must be such that, when one of the quantities is given, the other shall be determined by a linear equation. It follows that λ and μ must be related by the equation

$$A\lambda\mu + B\lambda + C\mu + D = 0,$$

where the ratios of the coefficients A, B, C, D shall depend, to some extent, upon δ. If, for instance, λ' and μ' be a given pair of parameters belonging to sets at the required intervene, then

$$A\lambda'\mu' + B\lambda' + C\mu' + D = 0,$$

by which the disposable coefficients in the homographic equation are reduced to two.

The converse of Axiom II., though generally true, is not universally so. It will, of course, generally happen that when two objects coincide their intervene is zero. But on every range two objects can be found, each of which is truly to be regarded as two coincident objects of which the intervene is not zero.

Let us, for instance, make $\lambda = \mu$ in the above equation; then we have

$$A\lambda^2 + (B + C)\lambda + D = 0.$$

This equation has, of course, two roots, each of which points out an object of critical significance on the range. We shall denote these objects by O and O'. Each of them consists of a pair of objects which, though actually coincident, have the intervene δ. The fundamental property of O and O' is thus demonstrated.

Let X be any object on the range; then (Axiom I.)

$$XO + \delta = XO;$$

and as δ is not zero, we have

$$XO = \text{infinity}.$$

Therefore every object on the range is at an infinite intervene from O. A similar remark may be made with respect to O'; and hence we learn that *the two objects, O and O', are at infinity.*

We assume, in Axiom III., that there are not to be more than two objects on the range at infinity: these are, of couse, O and O'. We must, therefore, be conducted to the same two objects at infinity, whatever be the value of the intervene δ, from which we started.* We thus see that while the original coefficients A, B, C, D do undoubtedly contain δ, yet that δ does not affect the equation

$$A\lambda^2 + (B + C)\lambda + D = 0.$$

* I published in *Nature* some time ago an investigation of this function, in which I had overlooked this important point. My attention was kindly directed to it by a subsequent letter from Mr. F. J. M'Aulay.

It follows that $D \div A$ and $(B + C) \div A$ must both be independent of δ. We may therefore make

$$A = A'; \quad B = B' + \Delta; \quad C = C' - \Delta; \quad D = D';$$

and thus the homographic equation becomes

$$A'\lambda\mu + (B' + \Delta)\lambda + (C' - \Delta)\mu + D' = 0,$$

where Δ is the only quantity which involves δ.

The equation can receive a much simpler form by taking the infinite objects as the two originating objects from which the range was determined. In this case the equation

$$A\lambda^2 + (B + C)\lambda + D = 0$$

must have as roots $\lambda = 0$ and $\lambda = \infty$, and therefore

$$A = 0; \quad D = 0,$$

hence the homographic equation reduces to

$$B\lambda + C\mu = 0;$$

since $B \div C$ is a function of the intervene δ, we may say, conversely, that

$$\delta = F\left(\frac{\lambda}{\mu}\right).$$

We have, however, already learned that the intervene is to have the form

$$\phi(\lambda) - \phi(\mu).$$

Now we find that it can also be expressed, with perfect generality, in the form

$$F\left(\frac{\lambda}{\mu}\right).$$

It follows that these two expressions must be equal, so that

$$\phi(\lambda) - \phi(\mu) = F\left(\frac{\lambda}{\mu}\right).$$

In this equation the particular value of δ does not appear, nor is it even implied. The formula must therefore represent an identical result true for all values of λ and all values of μ.

We may, therefore, differentiate the formula with respect both to λ and to μ, and thus we obtain

$$\phi'(\lambda) = \frac{1}{\mu} F'\left(\frac{\lambda}{\mu}\right)$$

$$-\phi'(\mu) = -\frac{\lambda^2}{\mu} F'\left(\frac{\lambda}{\mu}\right),$$

whence we deduce

$$\lambda\,\phi'(\lambda) = \mu\,\phi'(\mu).$$

As λ and μ are perfectly independent, this equation can only subsist by assuming for ϕ a form such that

$$\lambda\,\phi'(\lambda) = H,$$

where H is independent of λ. Whence we obtain

$$\phi'(\lambda) = \frac{H}{\lambda},$$

and, by integration,

$$\phi(\lambda) = H \log \lambda + \text{constant}.$$

The intervene is now readily determined, for

$$\phi(\lambda) - \phi(\mu) = H \log \lambda - H \log \mu = H \log \frac{\lambda}{\mu}.$$

We therefore obtain the following important theorem :—

Let x_1, x_2, x_3, x_4, and y_1, y_2, y_3, y_4 be the two objects at infinity on a range, and let $x_1 + \lambda y_1$, $x_2 + \lambda y_2$, $x_3 + \lambda y_3$, $x_4 + \lambda y_4$, and $x_1 + \mu y_1$, $x_2 + \mu y_2$, $x_3 + \mu y_3$, $x_4 + \mu y_4$ be any two other objects on the range, then their intervene will be expressed by

$$H(\log \lambda - \log \mu),$$

where H is a constant depending upon the adopted units of measurement.

§ 6. **A more General Investigation.**—Although the above is the shortest method of investigating the function, yet it will be useful to obtain the expression for the intervene in a rather more general manner.

The equation for λ and μ, corresponding to objects on a range at the intervene δ, is

$$A\lambda\mu + B\lambda + C\mu + D = 0.$$

Let λ' and λ'' be the two roots of this equation when μ is made equal to λ. It follows that what we have just written may be expressed thus:—

$$\lambda\mu + \lambda(\theta - \tfrac{1}{2}\lambda' - \tfrac{1}{2}\lambda'') + \mu(-\theta - \tfrac{1}{2}\lambda' - \tfrac{1}{2}\lambda'') + \lambda'\lambda'' = 0.$$

For, if $\lambda = \mu$, this is satisfied by either λ' or λ'', while θ disappears. θ is, of course, a function of the intervene, and it is only through θ that the intervene comes into the equation at all. By solving for θ, we find

$$\theta = \frac{\lambda\mu - \frac{1}{2}(\lambda' + \lambda'')(\lambda + \mu) + \lambda'\lambda''}{\mu - \lambda}.$$

The intervene itself is $F(\theta)$, where F expresses some function; and, accordingly,

$$\phi(\lambda) - \phi(\mu) = F(\theta).$$

When we substitute in this expression the value of θ given above, we have an identity which is quite independent of the particular δ. We must, therefore, determine the functions so that this equation shall remain true for all values of λ, and all values of μ. The formulæ must therefore be true when differentiated—

$$\frac{d\theta}{d\lambda} = \frac{(\mu - \lambda')(\mu - \lambda'')}{(\mu - \lambda)^2}, \quad \frac{d\theta}{d\mu} = \frac{(\lambda - \lambda')(\lambda'' - \lambda)}{(\mu - \lambda)^2},$$

$$\phi'(\lambda) = \frac{dF}{d\theta} \cdot \frac{d\theta}{d\lambda}, \qquad -\phi'\mu = \frac{dF}{d\theta} \cdot \frac{d\theta}{d\mu};$$

whence,

$$\frac{\phi'(\lambda)}{\phi'(\mu)} = \frac{(\mu - \lambda')(\mu - \lambda'')}{(\lambda - \lambda')(\lambda - \lambda'')},$$

or

$$(\lambda - \lambda')(\lambda - \lambda'')\phi'(\lambda) = (\mu - \lambda')(\mu - \lambda'')\phi'(\mu),$$

which has the form

$$\psi(\lambda) = \psi(\mu).$$

Considering the complete independence of both λ and μ, this equation requires that each of its members be independent alike of λ and μ. We shall denote it by $H(\lambda' - \lambda'')$; whence

$$(\lambda - \lambda')(\lambda - \lambda'')\phi'(\lambda) = H(\lambda' - \lambda''),$$

or,
$$\phi'(\lambda) = \frac{H(\lambda' - \lambda'')}{(\lambda - \lambda')(\lambda - \lambda'')}$$
$$= H\left(\frac{1}{\lambda - \lambda'} - \frac{1}{\lambda - \lambda''}\right);$$

whence, denoting the arbitrary constant by C,

$$\phi(\lambda) = H[\log(\lambda - \lambda') - \log(\lambda - \lambda'')] + C;$$

similarly,

$$\phi(\mu) = H[\log(\mu - \lambda') - \log(\mu - \lambda'')] + C;$$

and, finally, we have for the intervene, or $\phi(\lambda) - \phi(\mu)$, the expression,

$$\delta = H \log \frac{(\lambda - \lambda)(\mu - \lambda'')}{(\lambda - \lambda'')(\mu - \lambda')}.$$

This expression discloses the intervene as the logarithm of a certain anharmonic ratio.

We may here note how the obscurity referred to in the introduction is likely to arise in the mind of one who is approaching this theory for the first time. No doubt we find the intervene to be the logarithm of an anharmonic ratio of four quantities, but these quantities are not distances nor are they quantities homologous with the intervene. They are simply numerical. The four numbers, λ, μ, λ', λ'' are merely introduced to define four objects, one of them being,

$$x_1 + \lambda y, \quad x_2 + \lambda y_2, \quad x_3 + \lambda y_3, \quad x_4 + \lambda y_4,$$

and the others are obtained by replacing λ by μ, λ', λ'', respectively. All we assert is, that if we choose to call the two objects defined by λ' and λ'' the objects at infinity, and that if we desire the intervene between the objects λ and μ to possess the properties that we have already specified, then the only function possible will be the logarithm of the anharmonic ratio of these four numbers.

The word anharmonic is ordinarily applied in describing a certain function of four collinear points. In the more general sense, in which we are at this moment using the word, it does not relate to any geometrical or special

relation whatever; it is a purely arithmetical function of four abstract numbers.

We may also note that the relation between θ, in the equation at the commencement of this article, and the intervene δ is given by the equation

$$\theta = \tfrac{1}{2}(\lambda'' - \lambda')\frac{e^{\frac{\delta}{H}} + 1}{e^{\frac{\delta}{H}} - 1},$$

and the expression of the intervene as a function of θ; that is, the expression $F(\theta)$ is

$$\delta = H \log \frac{\theta + \frac{1}{2}(\lambda'' - \lambda')}{\theta - \frac{1}{2}(\lambda'' - \lambda')}.$$

III.—ON THE INTERVENE BETWEEN TWO OBJECTS IN AN EXTENT.

§ 7. **On the Infinite Objects in an Extent.**—On each range of the extent there will be two objects at infinity, by the aid of which the intervene between every two other objects on that range is to be ascertained. We are now to study the distribution of these infinite objects over the extent. Taking any range and one of its infinite objects, O, construct any other range in the same extent containing O as an object. This second range will also have two infinite objects. Is O to be one of them? Here we add another attribute to our, as yet, immature conception of the intervene.

In Euclidian space we cannot arrive at infinity except we take an infinitely long journey. This is because the point at infinity on one straight line is also the point at infinity on any other straight line passing through it. Were this not the case, then a finite journey to infinity could be taken by travelling along the two sides of a triangle in preference to the direct route *via* the third side. To develop the analogy between the conception of intervene and that of Euclidian distance, we therefore assume (in Axiom IV.) that an infinite object has an infinite intervene with every other object of the content. In consequence of this we have the general result, that—

If O be an infinite object on one range, then it is an infinite object on every one of the ranges diverging from O.

The necessity for this assumption is made clear by the following consideration :—Suppose that O were an infinite object on one range containing the object A, but were not an infinite object on another range OB, diverging also from O; then, although the direct intervene OA is infinite, yet the intervenes from A to B and from B to O would be both finite. The only escape is by the assumption we have just italicised. Otherwise infinity could be reached by two journeys, each of finite intervene.

Take any infinite object O. Construct a series of ranges in the extent, each containing O. Each of these ranges will have another infinite object, O_1, O_2, O_3, &c. The values x_1, x_2, x_3, which define O_1, O_2, O_3, &c., must fulfil some general condition, which we may express thus :

$$f(x_1, x_2, x_3) = 0.$$

Form a range through O_1 and O_2. There must be two infinite objects on this range, and of course all other objects thereon will be defined by a linear equation $L = 0$ in x_1, x_2, x_3.

Every object satisfying the condition $f(x_1, x_2, x_3)$ is infinite, and therefore all the values of x_1, x_2, x_3 common to the two equations $L = 0$ and $f(x_1, x_2, x_3) = 0$ must denote infinite objects. But we have already seen that there are only two infinite objects on one range; therefore there can be only two systems of values common to the two equations. In other words, $f(x_1, x_2, x_3)$ must be an algebraical function of the second degree. There can be no infinite object except those so conditioned; for, suppose that S were one, then any range through S would have two objects in common with f, and thus there would be three infinite objects on one range, which is impossible.

Hence we deduce the following important result :—

All the infinite objects in an extent lie on a range of the second degree.

We thus see that every range in an extent will have two objects in common with the infinite range of the second order. These are, of course, the two infinite objects on the range.

IV.—ON THE INTERVENE BETWEEN ANY TWO OBJECTS IN A CONTENT.

§ 8. **On the Periodic Term in the Complete Expression of the Intervene.**—We have found for the intervene the general expression

$$H(\log \lambda - \log \mu).$$

We may, however, write instead of λ,

$$(\cos 2n\pi + i \sin 2n\pi)\,\lambda,$$

where n is any integer; but this equals

$$e^{2in\pi};$$

hence,

$$\log \lambda = 2in\pi + \log \lambda;$$

and, consequently, the intervene is indeterminate to the extent of any number of integral multiples of

$$2iH\pi.$$

The expression just written is the intervene between any object and the same object, if we proceed round the entire circumference of the range. We may call it, in brief, the *circuit* of the range.

The intervene between the sets λ and $-\lambda$ is

$$iH\pi.$$

Nor is this inconsistent with the fact that $\lambda =$ zero denotes two coincident objects, as does also $\lambda = \pm$ infinity. In each of these cases the coincident objects are at infinity, and the intervene between two objects which coalesce into one of the objects at infinity has an indeterminate value, and may thus, of course, be $iH\pi$, as well as anything else.

§ 9. **Intervenes on Different Ranges in a Content.**—Let us suppose any two ranges whatever. There are an infinite number of objects on one range, and an infinite number on the other. The well-known analogies of homographic systems on rays in space lead us to inquire whether the several objects on the two ranges can be correlated homographically. Each object

in either system is to definitely correspond with a single object in the other system.

We determine an object on a range by its appropriate λ. Let the corresponding object on the other range be defined by λ'. The necessary conditions of homography demand that for one λ there shall be one λ', and *vice versa*. Compliance with this requires that λ and λ' shall be related by an equation of the form

$$P\lambda\lambda' + Q\lambda + R\lambda' + S = 0.$$

Let λ_1, λ_2, λ_3, λ_4 be any four values of λ, and let λ_1', λ_2', λ_3', λ_4' be the corresponding four values of λ', then, by substitution in the equation just written, and elimination of P, Q, R, S, it follows that

$$\frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} \times \frac{\lambda_2 - \lambda_4}{\lambda_1 - \lambda_4} = \frac{\lambda_1' - \lambda_3'}{\lambda_2' - \lambda_3'} \times \frac{\lambda_2' - \lambda_4'}{\lambda_1' - \lambda_4'}.$$

We now introduce the following important definition :—

By the expression, anharmonic ratio of four objects on a range, is meant the anharmonic ratio of the four values of the numerical parameter by which the objects are indicated.

We are thus enabled to enunciate the following theorem :—

When the objects on two ranges are ordered homographically, the anharmonic ratio of any four objects on one range equals the anharmonic ratio of their four correspondents on the other.

Three pairs of correspondents can be chosen arbitrarily, and then the equation last given will indicate the relation between every other λ and its corresponding λ'.

Among the different homographic systems there is one of special importance. It is that in which the intervene between any two objects in one range equals that between their correspondents in the other. But this homography is only possible when a critical condition is fulfilled.

In the first place, an infinite object on one range must have, as its correspondent, an infinite object on the other. For if X be an infinite object

on one range, it has an infinite intervene with every other object on that range (Axiom IV.); therefore X', the correspondent of X, must have an infinite intervene with every other object on the second range.

If, then, X and Y are the infinite objects on one range, and X' and Y' the infinite objects on the other, and if A and B be two arbitrary objects on the first range, and A' and B' their correspondents on the other; then, using the accustomed notation for anharmonic ratio,

$$(ABXY) = (A'B'X'Y').$$

But, if H' be the factor for the second range, which H is for the first, we have, since the intervenes are equal,

$$H \log (ABXY) = H' \log (A'B'X'Y');$$

and, since the anharmonic ratios are equal, we obtain

$$H = H'.$$

If, then, it be possible to order two homographic systems of objects, so that the intervene between any two is equal to that between their correspondents, we must have H and H' equal, and conversely, when H and H' are equal, then equi-intervene homography is possible.

We have therefore assumed Axiom V., § 4, which we have now seen to be equivalent to the assumption that the metric constant H is to be the same for every range of the content.

Nor is there anything in Axiom V. which constitutes it a merely gratuitous or fantastic assumption. Its propriety will be admitted when we reduce our generalized conceptions to Euclidian space. It is an obvious notion that any two straight lines in space can have their several points so correlated that the distance between a pair on one line is the same as that between their correspondents on the other. In fact, this merely amounts to the statement that a straight line marked in any way can be conveyed, marks and all, into a different situation, or that a foot-rule will not change the length of its inches *because* it is carried about in its owner's pocket.

In a similar, but more general manner, we desire to have it possible, on any two ranges, to mark out systems of corresponding objects, such that

the intervene between each pair of objects shall be equal to that between their correspondents. We have shown in this Article that such an arrangement is possible when, and only when, the property v is postulated. We may speak of such a pair of ranges as *equally graduated.*

The conception of *rigidity* involves the notion that it shall be possible to displace a system of points such that the distance between every pair of points in their original position equals that between the same pair after the displacement. We desire to have corresponding notions in the present Theory, which will only be possible when we have taken such a special view of the nature of the intervene as is implied in Axiom v.

§ 10. **Another Investigation of the possibility of equally Graduated Ranges.**—The importance of the subject in the last Article is so great in the present Theory that I here give it from a different point of view.

Taking the infinite objects on a range as the originating sets, we have on the first range for the intervene between the sets λ and α,

$$H(\log\lambda - \log\alpha);$$

and for the second range for the intervene between μ and β, we have

$$H'(\log\mu - \log\beta).$$

Regarding α and β as fixed, and λ and μ as defining a pair of correlative objects, we get, as the relation between λ and μ for equally graduated ranges,

$$H\log\frac{\lambda}{\alpha} = H\log\frac{\mu}{\beta};$$

whence the relation between λ and μ is thus given :

$$\frac{\lambda}{a} = \left(\frac{\mu}{b}\right)^{\frac{H'}{H}}.$$

When this is the case, there are several values of λ corresponding to one value of μ, which may be thus found :—Let m be any integer; then, in the usual manner,

$$\frac{\lambda}{\alpha} = \left(\frac{\mu}{\beta}\right)^{\frac{H'}{H}}\left(\cos 2m\frac{H'}{H}\pi + i\sin 2m\frac{H'}{H}\pi\right);$$

and therefore

$$\frac{\lambda}{\alpha} = \left(\frac{\mu}{\beta}\right)^{\frac{H'}{H}},$$

or

$$= \left(\frac{\mu}{\beta}\right)^{\frac{H'}{H}} \left(\cos \frac{2H'}{H}\pi + i \sin \frac{2H'}{H}\pi\right),$$

or

$$= \left(\frac{\mu}{\beta}\right)^{\frac{H'}{H}} \left(\cos \frac{4H'}{H}\pi + i \sin \frac{4H'}{H}\pi\right),$$

&c. &c. &c.

We see that the correspondence between λ and μ cannot be of the homographic, or one-to-one character, unless

$$H = H'.$$

The comprehension of the subject may be facilitated by looking at it in the following manner:—

The intervene between any two objects on one range is, of course, ambiguous, to the extent of any integral number of the circuits on that range. Let C and C' be the circuits, and let δ be an intervene between two objects on the second range. If we try to determine two objects, α and λ, on the first range that shall have an intervene δ, we must also have another object λ', such that its intervene from α is $\delta + C'$. Similarly, there must be another object λ'' with the intervene $\delta + 2C'$, &c. It is therefore impossible to have a single object at the intervene $\delta + mC'$ from α, unless it happened that

$$C = C',$$

or that

$$H = H'.$$

Thus, again, are we led to the conclusion that ranges cannot be equally graduated unless their circuits are the same.

The circuits on every range in the content being now taken to be equal, we can assume for the circuit any value we please. There are great advan-

tages in so choosing our units that the circuit shall be π; but we have as its expression,

$$2i\,H\pi;$$

whence we deduce

$$2iH = 1.$$

§ 11. **On the Infinite Objects in the Content.**—Let a_1, a_2, a_3, a_4 be four objects which are not connected by any numerical relation. Let x_1, x_2, x_3, x_4 be four arbitrary numerical quantities. Then the group of objects typified by

$$x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4,$$

where x_1, x_2, x_3, x_4 are variable, constitute the content. Certain objects in the content are infinite, and it is proposed to determine the conditions imposed on x_1, x_2, x_3, x_4 when they indicate one of these. If an object O be infinite, then every range through that object will have one other infinite object. Let these be O_1, O_2, &c. These several objects will conform with the condition,

$$f(x_1, x_2, x_3, x_4) = 0$$

Every infinite object in the content must satisfy this equation; and, conversely, every object so circumstanced is infinite.

Two linear equations in x_1, x_2, x_3, x_4 determine a range, and the simultaneous solution of these equations with

$$f(x_1, x_2, x_3, x_4) = 0$$

gives the infinite objects on that range—but there can only be two—and hence we have the following important theorem:—

The co-ordinates x_1, x_2, x_3, x_4 of the infinite objects in a content satisfy an homogeneous equation of the second degree.

We denote this equation by

$$U = 0.$$

V.—ON THE THEORY OF DEPARTURE.

§ 12. **The Star.**—The principle of Duality enters in a significant manner into the present theory. We have hitherto taken the object as the primary element, and from two objects we have obtained the range. Confining our attention, for the present, to the same *extent*, we are now to make the range the fundamental element, while the object originates from two ranges.

Let the symbols a_1 and a_2 denote ranges, and

$$x_1 a_1 + x_2 a_2,$$

where x_1 and x_2 are numerical magnitudes, will denote a new range. If $x_1 \div x_2$ vary, we shall have several ranges; and the entire group of ranges produced in this way we call a *star*.

§ 13. **The Departure.**—In ordinary geometry the most important function of a pair of rays in a pencil is that which expresses their inclination. We have now to create, for our generalized conceptions, a function of two ranges in a star which shall be homologous with the notion of ordinary angular magnitude.

We shall call this function the *Departure*. Its form is to be determined by the properties that we wish it to possess. In the investigation of the departure between two ranges, we shall follow steps parallel to those which determined the intervene between two objects.

If OP, OQ, OR be three rays in an ordinary plane diverging from O, then

$$\angle POQ + \angle QOR = \angle POR.$$

In general the angle between two rays is not zero unless the rays are coincident; but this statement ceases to be true when the vertex of the pencil is at infinity. In this case, however, the angle between every pair of rays in the pencil is zero.

Every plane pencil has two rays (*i. e.* those to the circular points at infinity), which make an infinite angle with every other ray.

§ 14. **Second Group of Axioms of the Content.**—We desire to construct a departure function which shall possess the following properties:—

VI. If three ranges, P, Q, R, in a star, be ordered in ascending $x_2 \div x_1$, and if the departure between two ranges, for example, P and Q, be expressed by PQ, then

$$PQ + QR = PR.$$

VII. The departure between two ranges cannot be zero unless the ranges are coincident, or unless the departure between every pair of ranges in that star is also zero.

VIII. Of the ranges in a star, two (distinct or coincident) are at infinity, *i. e.* have each an infinite departure from all the remainder.

IX. An infinite range has an infinite departure, not only with every range in its star, but with *every* range in the extent.

X. If the several ranges in one star correspond one to one with the several ranges in another, and if the two infinite ranges in one star have as their correspondents the infinite ranges in the other; then the departure between any two ranges in one star is equal to that between the two corresponding ranges in the other.

§ 15. **The Form of the Departure Function.**—The analogy of these several axioms to those which have guided us to the discovery of the intervene, show that the investigation for the function of departure will be conducted precisely as that of the intervene has been; accordingly, we need not repeat the several steps of the investigation, but enunciate the general result, as follows:—

Let x_1, x_2, and y_1, y_2 be the co-ordinates of any two ranges in a star, and let λ_1, λ_2, and μ_1, μ_2 be the co-ordinates of the two infinite ranges in that star. Then the departure between x and y is

$$H \log \frac{(x_1\lambda_2 - x_2\lambda_1)(y_1\mu_2 - y_2\mu_1)}{(y_1\lambda_2 - y_2\lambda_1)(x_1\mu_2 - x_2\mu_1)}.$$

§ 16. **On the Arrangement of the Infinite Ranges.**—Every star in the extent will have two infinite ranges, and we have now to see how these

several infinite ranges in the extent can be compendiously organized into a whole.

To aid in this we have assumed Axiom IX., the effect of which is to render the following statement true. Let several objects on a range, O, be the vertices of a corresponding number of stars. If O be an infinite range in *any* one of the stars, then it is so in *every* one.

Let a_1, a_2, a_3 be any three ranges in an extent. Then every range in the same extent can be expressed by

$$x_1a_1 + x_2a_2 + x_3a_3,$$

where x_1, x_2, x_3 are the three co-ordinates of the range. It is required to determine the relation between x_1, x_2, x_3 if this range be infinite.

An object L will be defined by an equation of the form, where L_1, L_2, L_3 are numerical,

$$L_1x_1 + L_2x_2 + L_3x_3 = 0,$$

or any two sets of values, x_1, x_2, x_3, which satisfy this equation, will determine a pair of ranges which have the required object in common.

Let the relation between the co-ordinates of an infinite range be

$$f(x_1, x_2, x_3) = 0\,;$$

then the infinite ranges in the star, whose vertex is the object L, will be defined by co-ordinates obtained from the simultaneous solution of

$$L_1x_1 + L_2x_2 + L_3x_3 = 0,$$

$$f(x_1, x_2, x_3) = 0.$$

But there can only be two such ranges; and, accordingly, the latter of these equations must be of the second degree. We hence deduce the following important result:—

All the infinite ranges in an extent may be represented by the different values of $x_1a_1 + x_2a_2 + x_3a_3$, where a_1, a_2, a_3 are any arbitrary ranges, and where x_1, x_2, x_3 are numerical magnitudes satisfying one homogeneous equation of the second degree.

§ 17. **Relations between Departure and Intervene.**—With reference to the theory of Departure, we thus see that there is a system of critical

ranges, and a system of critical objects in each extent. Every critical range has an infinite departure from every other range. Every critical object is the vertex of a star of which the departure between every pair of ranges is zero. It will be remembered that in the theory of the intervene we were conducted to the knowledge that every extent contained a system of objects and ranges, critical with regard to the intervene. Each critical object had an infinite intervene with every other object in the extent. Each critical range possessed the property that the intervene between any two objects thereon is zero. There are, thus, objects and ranges critical with respect to the intervene. There are also objects and ranges critical with respect to the departure. But nothing that we have hitherto assumed will entitle us to draw any inference as to the connexion, much less as to the actual identity, between the critical systems related to intervene, and those related to departure. We have already assumed five properties for the intervene, and five like properties for the departure. These are, in fact, the axioms by which alone the functions of intervene and of departure could be constructed. But another axiom of quite a distinct type has now to be introduced.

There are objects of infinite intervene, and objects of zero departure. There are ranges of infinite departure, and ranges of zero intervene. A range *generally* contains two objects of infinite intervene, and two of zero departure. A star *generally* contains two ranges of infinite departure, and two ranges of zero intervene. On a range of zero intervene the two objects of infinite intervene coalesce, and their intervene from other objects on the range becomes indeterminate. In a star of zero departure the two ranges of infinite departure coalesce, and their departure from other ranges in the same star becomes indeterminate. We have thus the following statement:—

On a range of zero intervene, the intervene between every pair of objects is zero, except where one particular object is involved, in which case the intervene is indeterminate.

In a star of zero departure, the departure between every pair of ranges is zero, except where one particular range is involved, in which case the departure is indeterminate.

Remembering that the existence of zero intervene between every pair of objects on a range is a consequence of the coincidence of the two objects of infinite intervene on that range, we have the following result:—

The range through two consecutive objects of infinite intervene is a range of zero intervene.

And, similarly, we have the following:—

The object common to two consecutive ranges of infinite departure is the vertex of a star of zero departure.

The new axiom to be now introduced will be formed as the others have been by generalization from the ordinary conceptions of geometry. In that geometry we have two different aspects in which the phenomenon of parallelism may be presented. Two non-coincident lines are parallel when the angle between them is zero, or when their intersection is at an infinite distance. Without entering into any statement about parallel lines, we may simply say, that when two different straight lines are inclined at the angle zero, their point of intersection is at infinity. Generalizing this proposition, we assume the following axiom or property, which we desire that our systems of measurement shall possess.

§ 18. **The Eleventh Axiom of the Content.**—This axiom, which is the first to bring together the notions of intervene and of departure, is thus stated:—

XI. *If two ranges in the same extent have zero departure, their common object will be at infinity, and conversely.*

The vertex of every star of zero departure will thus be at infinity, and hence we deduce the important result that all the objects of infinite intervene are also objects of zero departure, and conversely. Thus we see that the two systems of critical objects in the extent coalesce into a single system in consequence of the assumption in Axiom XI.

Each consecutive pair of critical objects determine a range, which is a range of infinite departure as well as of zero intervene.

The expression *infinite objects* will then denote objects which possess the double property of having, in general, an infinite intervene from other objects in the extent, and of being also the vertices of stars of zero departure.

The expression *infinite ranges* will denote ranges which possess the double property of having, in general, an infinite departure with all other ranges, and which consist of objects, the intervene between any pair of which is, in general, zero.

§ 19. **Professor Heath's Theorems.**—There is still one more point to be decided. The measurement of departure, like that of intervene, is expressed by the product of a numerical factor with the logarithm of an anharmonic ratio. This factor is H for the intervene. Let us call it H' for the departure. What is to be the relation between H and H'? Here the analogy of geometry is illusory; for, owing to the coincidence between the points of infinity on a straight line, H has to be made infinite in ordinary geometry, while H' must be finite. But in the present more general theory H is finite, and we have found much convenience derived from making it equal to $-\frac{i}{2}$, for then the entire circuit of any range is π. We now stipulate that H' is also to be $-\frac{i}{2}$. The circuit of a star will then be π also.

With this assumption the theory of the metrics of an extent admits of the remarkable development discovered by Professor Heath. His results, stated in our nomenclature, are embodied in the following proposition:—

Let x, y, z be any three objects. Let a, b, c denote the intervenes between y and z, z and x, x and y, respectively. Let the departure between the ranges from x to y and x to z be denoted by A, from y to z and y to x be denoted by B, and from z to x and z to y be denoted by C. Then,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c};$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Thus the formulæ of spherical trigonometry are generally applicable throughout the extent. This is a truly astonishing theorem.

VI.—ON THE REPRESENTATION OF THE OBJECTS IN A CONTENT BY THE POINTS IN SPACE.

§ 20. **Introductory.**—The several objects in a content are each completely specified when the four numbers, x_1, x_2, x_3, x_4, corresponding to each are known. It is only the ratios of these numbers that are significant. We may hence take them to be the four quadriplanar co-ordinates of a point in space. We are thus led to the construction of a system of one-to-one correspondence between the several points of an Euclidian space, and the several objects of a content. The first proposition is the following:—

One object in a content has for its correspondent one point in space, and one point in space corresponds to one object in the content.

Let two objects in the content be

$$x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4,$$

and

$$y_1a_1 + y_2a_2 + y_3a_3 + y_4a_4;$$

then the range defined by these objects is obtained by giving all values to λ in the expression

$$(x_1 + \lambda y_1)\, a_1 + (x_2 + \lambda y_2)\, a_2 + (x_3 + \lambda y_3)\, a_3 + (x_4 + \lambda y_4)\, a_4.$$

The system of points correlated to these objects have, as their co-ordinates,

$$x_1 + \lambda y_1, \quad x_2 + \lambda y_2, \quad x_3 + \lambda y_3, \quad x_4 + \lambda y_4.$$

Hence we deduce the result, that

The several objects on a range correspond one to one with the several points on a straight line.

If we consider three objects, namely,

$$x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4,$$

$$y_1a_1 + y_2a_2 + y_3a_3 + y_4a_4,$$

$$z_1a_1 + z_2a_2 + z_3a_3 + z_4a_4,$$

then an *extent* consists of the several objects obtained by giving all values to λ and μ in the expression,

$$(x_1 + \lambda y_1 + \mu z_1)\, a_1 + (x_2 + \lambda y_2 + \mu z_2)\, a_2 + (x_3 + \lambda y_3 + \mu z_3)\, a_3 + (x_4 + \lambda y_4 + \mu z_4)\, a_4.$$

The several points correlated to these objects have, as their co-ordinates,

$$x_1 + \lambda y_1 + \mu z_1, \quad x_2 + \lambda y_2 + \mu z_2, \quad x_3 + \lambda y_3 + \mu z_3, \quad x_4 + \lambda y_4 + \mu z_4,$$

and hence we obtain the result that—

The several objects in an extent correspond one to one with the several points in a plane.

Since the objects at infinity are obtained by taking values of x_1, x_2, x_3, x_4, which satisfy a quadric equation, we find that—

The several objects at infinity in the content correspond with the several points of a quadric surface.

This surface we shall call the *infinite quadric.*

Remembering the definition of the anharmonic ratio of four objects on a range (§ 6), we obtain the following theorem :—

The anharmonic ratio of four objects on a range equals the anharmonic ratio of their four corresponding points on a straight line.

We hence deduce the following important result, that—

The intervene between any two objects is proportional to the logarithm of the anharmonic ratio in which the straight line joining the corresponding points is divided by the infinite quadric.

We similarly find that—

The departure between any two ranges in the same extent is proportional to the logarithms of the anharmonic ratio of the pencil formed by their two corresponding straight lines, and the two tangents in the same plane from their intersection to the infinite quadric.

§ 21. **Poles and Polars.**—The point x_1, x_2, x_3, x_4 has for its polar, with regard to the infinite quadric, the plane,

$$x_1 \frac{dU}{dx_1} + x_2 \frac{dU}{dx_2} + x_3 \frac{dU}{dx_3} + x_4 \frac{dU}{dx_4} = 0.$$

Thus we see that an object corresponding to the point will have a polar extent corresponding to the polar of that point with regard to the infinite quadric. The following property of poles and polars follows almost immediately.

The intervene from an object to any object in its polar extent is equal to $\frac{\pi}{2}$.

We have hitherto spoken of the departure between a pair of ranges which have a common object: we now introduce the notion of the departure between a pair of extents by the following definition:—

The departure between a pair of extents is equal to the intervene between their poles.

VII.—ON THE HOMOGRAPHIC TRANSFORMATION OF THE CONTENT.

§ 22. **General Principle.**—In our further study of the theory of the content we shall employ, instead of the objects themselves, their corresponding points in ordinary space. All the phenomena of the content can be completely investigated in this way. Objects, ranges, and extents, we are to replace by points, straight lines, and planes. Intervenes are to be measured, not, indeed, as distances, but as logarithms of certain anharmonic ratios obtained by ordinary distance measurement. Departures are to be measured, not, indeed, as angles, but as logarithms of anharmonic ratios of certain pencils obtained by ordinary angular measurement.

The theory of the content is no doubt equivalent to, or identical with the theory of so-called non-Euclidian geometry, or geometry of elliptic space. I have indeed, myself, frequently used these latter expressions, but am glad to take this opportunity of renouncing them. I have found the associations they suggest very misleading; and, in fact, it was only by employing some such conceptions and terminology as that used in the present Paper that I succeeded in understanding the subject. *It is especially needful to avoid confusing the sign with the thing signified.* The space-points are only the signs of the objects—they are not the objects themselves—and, if we are not careful to preserve this distinction, obscurity will arise about the intervene. For we have all through this Paper sought to develop the conception of

the intervene as the general conception of distance between two objects; but this is a totally different thing from the distance between their two corresponding points.

I now suppose the several objects of a content to be ordered in two homographic systems, A and B. Each object, X, in the content, regarded as belonging to the system A, will have another object, Y, corresponding thereto in the system B. The correspondence is to be simply of the one-to-one type. Each object in one system has one correspondent in the other system. But if X be regarded as belonging to the system B, its correspondent in A will not be Y, but some other object, Y'.

To investigate this correspondence we shall represent the objects by their correlated points in space. We take x_1, x_2, x_3, x_4 as the co-ordinates of a point corresponding to x, and y_1, y_2, y_3, y_4 as the co-ordinates of the point corresponding to y. We are then to have an unique correspondence between x and y, and we proceed to study the conditions necessary if this be complied with.

§ 23. **Deduction of the Equations of Transformation.**—All the points in a plane L, taken as x points, must have as their correspondents the points also of a plane; for, suppose that the correspondents formed a surface of the nth degree, then three planes will have three surfaces of the nth degree as their correspondents, and all their n^3 intersections regarded as points in the second system will have but the single intersection of the three planes as their correspondents in the first system. But this does not accord with the assumption that the correspondence is to be of the one-to-one type unless $n = 1$. Hence we see that to a plane of the first system will correspond a plane of the second system, and *vice versa.*

Let the plane in the second system be

$$A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_4 = 0.$$

If we seek its corresponding plane in the first system, we must substitute for y_1, y_2, y_3, y_4 the corresponding functions of x_1, x_2, x_3, x_4. Now, unless these are homogeneous linear expressions, we shall not find that this remains

a plane. Hence we see that the relations between x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 must be of the following type where (11), (12), &c., are constants:—

$$y_1 = (11)\,x_1 + (12)\,x_2 + (13)\,x_3 + (14)\,x_4,$$

$$y_2 = (21)\,x_1 + (22)\,x_2 + (23)\,x_3 + (24)\,x_4,$$

$$y_3 = (31)\,x_1 + (32)\,x_2 + (33)\,x_3 + (34)\,x_4,$$

$$y_4 = (41)\,x_1 + (42)\,x_2 + (43)\,x_3 + (44)\,x_4.$$

Such are the equations expressing the general homographic transformation of the objects of a content. From the general theory, however, we now proceed to specialize one particular kind of homographic transformation. It is suggested by the notion of a *displacement* in ordinary space. The displacement of a rigid system is only equivalent to a homographic transformation of all its points, conducted under the condition that the distance between every pair of points shall remain unaltered. In our extended conceptions we now study the possible homographic transformations of a content, conducted subject to the condition that the intervene between every pair of objects shall equal that between their correspondents.

§ 24. **On the Character of a Homographic Transformation which Conserves Intervene.**—In all investigations of this nature the behaviour of the infinite objects is especially instructive. In the present case it is easily shown that every object O infinite before the transformation must be infinite afterwards when moved to O'. For, let X be any object which is not infinite before the transformation, nor afterwards, when it becomes X'. Then, by hypothesis, the intervene OX is equal to $O'X'$; but OX is infinite, therefore $O'X'$ must be also infinite, so that either O' or X' is infinite; but, by hypothesis, X' is finite, therefore O' must be infinite, so that *in a homographic transformation which conserves intervene, each object infinite before the transformation remains infinite afterwards.*

It follows that in the space representation each point, X, representing an infinite object, and therefore lying on the infinite quadric $U = 0$ must, after transformation, be moved to a position y, which will also lie on the infinite quadric. Hence we obtain the following important result:—

In the space representation of a homographic transformation which conserves intervene, the infinite quadric $U = 0$ is merely displaced on itself.

A homographic transformation of the points in space will not, *in general*, permit any quadric to remain unchanged. A certain specialization of the constants (11), (12), &c., will be necessary. They must, in fact, satisfy a single condition, for which we shall presently find the expression.

In the general case there are, of course, four points in space, of which the positions are unaltered by the homographic transformation. They are found in the usual manner.

Assume

$$y_1 = \rho x_1\,; \quad y_2 = \rho x_2\,; \quad y_3 = \rho x_3\,; \quad y_4 = \rho x_4.$$

Then there will be four values of ρ, namely, the roots of the equation—

$$0 = \begin{vmatrix} (11)-\rho & (12) & (13) & (14) \\ (21) & (22)-\rho & (23) & (24) \\ (31) & (32) & (33)-\rho & (34) \\ (41) & (42) & (43) & (44)-\rho \end{vmatrix}.$$

We may call this the *critical equation*, as the character of the homographic transformation is to be decided by its aid. When ρ is made zero, this determinant is termed Δ.

The four values of ρ, namely, ρ_1, ρ_2, ρ_3, ρ_4 will each determine a quartet of co-ordinates appropriate to the four double points of the system. We shall denote one of them as follows:—

$$x_1', \quad x_2', \quad x_3', \quad x_4',$$

with two, three, and four dashes for the three others. These are the corners of a tetrahedron, of which one plane will be represented by the equation $X_1 = 0$, in which X_1 is the determinant—

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1' & x_2' & x_3' & x_4' \\ x_1'' & x_2'' & x_3'' & x_4'' \\ x_1''' & x_2''' & x_3''' & x_4''' \end{vmatrix}.$$

Multiplying this by the determinant Δ, it becomes, after a simple reduction, $\rho_1\rho_2\rho_3 Y_1$, where Y_1 is the determinant—

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ x_1' & x_2' & x_3' & x_4' \\ x_1'' & x_2'' & x_3'' & x_4'' \\ x_1''' & x_2''' & x_3''' & x_4''' \end{vmatrix}.$$

Remembering that $\Delta = \rho_1\rho_2\rho_3\rho_4$, we have, for the relation between X_1 and Y_1, X_2, and Y_2, &c., the equations,

$$Y_1 = \rho_1 X_1; \quad Y_2 = \rho_2 X_2; \quad Y_3 = \rho_3 X_3; \quad Y_4 = \rho_4 X_4.$$

The infinite quadric $U = 0$ is expressed in terms of x_1, x_2, x_3, x_4; but as the latter can be linearly expressed by X_1, X_2, X_3, X_4, it follows that the infinite quadric must admit of being expressed in the form,

$$f(X_1, \quad X_2, \quad X_3, \quad X_4) = 0.$$

This equation is to remain satisfied if each point be replaced by its correspondent, and therefore

$$f(Y_1, \quad Y_2, \quad Y_3, \quad Y_4) = 0,$$

We have to discover the form of f, from the knowledge that when we substitute $\rho_1 X_1$, $\rho_2 X_2$, $\rho_3 X_3$, $\rho_4 X_4$, for X_1, X_2, X_3, X_4, respectively, the equation remains unaltered.

Let the quadric be

$$A_{11}X_1^2 + \&c., + 2A_{12}X_1X_2 + \&c., = 0,$$

then, making the substitutions, and expressing the conditions that the quadrics shall be identical, we obtain,

$$\frac{A_{11}\rho_1^2}{A_{11}} = \frac{A_{22}\rho_2^2}{A_{22}} = \frac{A_{33}\rho_3^2}{A_{33}} = \frac{A_{44}\rho_4^2}{A_{44}} = \frac{A_{12}\rho_1\rho_2}{A_{12}} = \&c.$$

There are several solutions of the problem.

(1). Let all the coefficients, A_{12}, A_{22}, &c., be zero, except one, for example A_{11}, then the quadric becomes

$$X_1^2 = 0,$$

denoting, of course, merely the face of the tetrahedron.

(2). If all the co-efficients were zero, except A_{12}, then the quadric will be the pair of planes,

$$X_1 X_2 = 0.$$

(3). If all the coefficients, except two, be zero, there will be several cases, of which the first is where A_{11} and A_{22} remain, and the surface becomes the pair of planes,

$$A_{11}X_1^2 + A_{22}X_2^2 = 0,$$

with the condition,

$$\rho_1^2 = \rho_2^2.$$

(4). Or we may have A_{11} and A_{12} remaining, and the quadric is

$$A_{11}X_1^2 + A_{12}X_1X_2 = 0.$$

Here, again, the quadric is a pair of planes, provided that

$$\rho_1^2 = \rho_1\rho_2.$$

(5). Another case would be, where A_{12} and A_{13} alone remained. The quadric is then the pair of planes,

$$A_{12}X_1X_2 + A_{13}X_1X_3 = 0,$$

with the condition,

$$\rho_1\rho_2 = \rho_1\rho_3.$$

(6). Another of the binary forms would be, where A_{12} and A_{34} remained. The equation then is,

$$A_{12}X_1X_2 + A_{34}X_3X_4 = 0,$$

with the accompanying condition,

$$\rho_1\rho_2 = \rho_3\rho_4.$$

(7). We might also have retained A_{11} and A_{23}, and thus we should get

$$A_{11}X_1^2 + A_{23}X_2X_3,$$

with the condition,

$$\rho_1^2 = \rho_2\rho_3.$$

There could also be several invariable quadrics, consisting of three terms,

provided the values of ρ were subjected to two equations. Thus, for example, if

$$\rho_1^2 = \rho_2^2 = \rho_3\rho_4,$$

then the quadric,

$$A_{11}X_1^2 + A_{22}X_2^2 + 2A_{34}X_3X_4 = 0,$$

will remain unaltered.

If we made the values of ρ to satisfy three equations, then there could be quadrics with four terms, which remain unaltered by the transformation.

From among these various quadrics, we have to select one which answers our purpose; here again we have to resort to ordinary geometry for guidance.

In the displacement of a rigid system in ordinary space one ray remains unchanged, and so does every circular cylinder of which this ray is the axis. Thus we see that there is a whole family of cylinders which remain unchanged; and if U be one of these cylinders, and V another, then all the cylinders of the type $U + \lambda V$ are unaltered, the plane at infinity being of course merely an extreme member of the series. More generally these cylinders may be regarded as a special case of a system of cones with a common vertex; and more generally still we may say that a family of quadrics remains unchanged.

Reverting, then, to a space of which the several points correspond to the objects of a content, we find, that for every homographic transformation which corresponds to a displacement in ordinary geometry a *singly* infinite family of quadrics is to remain unchanged, and the infinite quadric itself is to form one member of this family. This being so, we need not consider any equation which contain more than two terms; for example, we may reject

$$A_{11}X_1^2 + A_{22}X_2^2 + 2A_{34}X_3X_4 = 0\,;$$

because, when the proper conditions among the roots are satisfied, then every surface of this type remains unaltered, but these would form a *doubly* infinite family.

We are, therefore, limited to the surfaces expressed by one or two terms, of which these are the seven types, 1–7. Of these, the only family including quadrics of a general type is (6),

$$A_{12}X_1X_2 + A_{34}X_3X_4 = 0:$$

all the others are either cones or the products of planes. If, therefore, the infinite quadric is to be a surface of a general type, we are restricted to an equation of the form just written.

What the infinite quadric is to be must form a matter for stipulation. It would be quite possible to construct a theory of intervene for which the infinite quadric was a cone. But we do not wish to delay over special cases: we want the general theory. If we desired to specially exclude the case of a cone, it would be sufficient to enunciate an axiom, to the effect that it shall not be possible for one object in the content to have an indeterminate intervene with *every* other object in the content. Were the infinite quadric a cone, the object corresponding to its vertex would have an intervene, $0 \div 0$, with respect to every other object in the content.

We thus find that the general equation to the infinite quadric must be one of the surfaces,

$$A_{12}X_1X_2 + A_{34}X_3X_4 = 0,$$

where the ratio $A_{12} \div A_{34}$ assumes some particular value. Every surface of this family remains unaltered by the homographic transformation, and the condition

$$\rho_1\rho_2 - \rho_3\rho_4 = 0,$$

must be satisfied.

Let us now suppose a range in the content submitted to this description of homographic transformation. Let P, Q be two objects on the range, and let X, Y be the two infinite objects thereon. This range will be transformed to a new position, and the objects will now be P', Q', X', Y'. Since infinite objects must remain infinite, it follows that X' and Y' must be infinite, as well as X and Y. Also, since homographic transformation does not alter anharmonic ratio, we have

$$(PQXY) = (P'Q'X'Y');$$

whence, by Axiom v., we see that the intervene from P to Q equals the intervene from P' to Q'; in other words, that *all* intervenes remain unchanged by this homographic transformation.

Heath's theorem obviously enables us to add the statement, that all departures are also unaltered.

Every homographic transformation which possesses these properties must satisfy a special condition in the coefficients. This may be found as follows:—Write the equation for ρ, viz.,

$$0 = \begin{vmatrix} (11)-\rho & (12) & (13) & (14) \\ (21) & (22)-\rho & (23) & (24) \\ (31) & (32) & (33)-\rho & (34) \\ (41) & (42) & (43) & (44)-\rho \end{vmatrix};$$

then the following symmetric function of the four roots ρ_1, ρ_2, ρ_3, ρ_4 must vanish:

$$(\rho_1\rho_2 - \rho_3\rho_4)(\rho_1\rho_3 - \rho_2\rho_4)(\rho_1\rho_4 - \rho_2\rho_3).$$

§ 25. **On the Geometrical Meaning of this Symmetric Function.**—We may write the family of quadrics thus:

$$X_1X_2 + \lambda X_3X_4 = 0.$$

All these quadrics have two common generators of each kind:

$$\left\{\begin{array}{c} X_1 = 0, \quad X_3 = 0 \\ \text{and} \\ X_2 = 0, \quad X_4 = 0 \end{array}\right. \qquad \left\{\begin{array}{c} X_1 = 0, \quad X_4 = 0 \\ \text{and} \\ X_2 = 0, \quad X_3 = 0. \end{array}\right.$$

For the rays $X_1 = 0$, $X_3 = 0$, and $X_1 = 0$, $X_4 = 0$, are both contained in the plane X_1, and therefore intersect, and, accordingly, belong to the opposed system of generators.

The geometrical meaning of the equation

$$\rho_1\rho_2 - \rho_3\rho_4 = 0$$

can be also shown.

The tetrahedron formed by the intersection of the two pairs of generators just referred to remains unaltered by the transformation. Any point on the edge, $X_1 = 0$, $X_3 = 0$, of which the co-ordinates are

$$0, \ X_2, \quad 0, \ X_4,$$

will be transformed to

$$0, \ \rho_2 X_2, \quad 0, \ \rho_4 X_4.$$

The theory will be more easily understood by Figure 1.

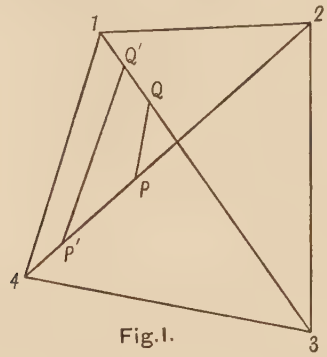


Fig. 1.

Let 1, 2, 3, 4 be the four corners of the tetrahedron. Let the transformation convey P to P' and Q to Q'. As P varies along the ray, so will P vary, and the two will describe homographic systems, of which 2 and 4 are the double points. In a similar way, Q and Q' will trace out homographic systems on the ray 1 3. We shall write the points on 2 4, in the order,

$$2,\ 4,\quad P,\ P'.$$

Through 2, the generator of the surface 2 3 can be drawn (1 2 is *not* a generator), and through 4 the generator 4 1 can be drawn (4 3 is not a generator); thus we have, for the corresponding order on 1 3,

$$3,\quad 1,\quad Q,\quad Q'.$$

Points.	Co-ordinates.				Points.	Co-ordinates.			
2	0	1	0	0	3	0	0	1	0
4	0	0	0	1	1	1	0	0	0
P	0	X_2	0	X_4	Q	X_1'	0	X_3'	0
P'	0	$\rho_2 X_2$	0	$\rho_4 X_4$	Q'	$\rho_1 X_1'$	0	$\rho_3 X_3'$	0

The anharmonic ratio of the first set is that of $0,\ \infty,\ \dfrac{X_4}{X_2},\ \dfrac{\rho_4}{\rho_2}\dfrac{X_4}{X_2}$.

,, ,, second ,, $\infty,\ 0,\ \dfrac{X_3}{X_1'},\ \dfrac{\rho_3}{\rho_1}\dfrac{X_3'}{X_1'}$;

but since $\rho_1\rho_2 = \rho_3\rho_4$,

then the anharmonic ratios are equal.

The theorem can otherwise be shown by drawing it as in Figure 2.

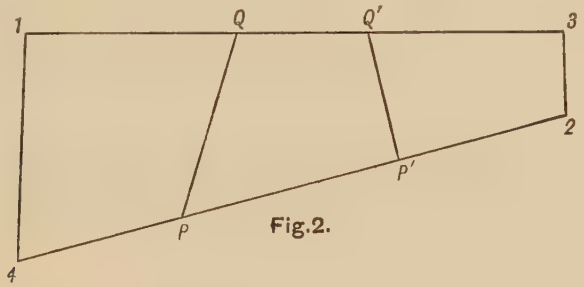


Fig.2.

1 4 and 3 2 are to be generators of the infinite quadric. This will show that 4 (and not 2) is the correspondent to 1, and that 2 (and not 4) is the correspondent to 3, and thus the statement of anharmonic equality,

$$(1\ QQ'\ 3) = (4\ PP'\ 2),$$

becomes perfectly definite.

1 2 and 3 4 are, of course, not generators: they are two conjugate polars of the infinite quadric.

We can now see the reason of the anharmonic equality. Let PQ be a generator of the infinite quadric, as is clearly possible, for 1, 3 and 2, 4 are both generators of the opposite system. Then, since a generator of the infinite quadric must remain thereon after the displacement, it will follow that $P'Q'$, to which PQ is displaced, must also be a generator; and thus we have four generators, 4 1, PQ, $P'Q'$, 2 3, on a hyperboloid of one system intersecting the two generators of another, and by the well-known property of the surface,

$$(1\ QQ'\ 3) = (4\ PP'\ 2).$$

We also see why the infinite quadric is only one of a family which remains unaltered. For, if PQ be a generator of *any* quadric through the tetrahedron, 1, 2, 3, 4; then, since P and Q are conveyed to P' and Q', and since the anharmonic equality holds, it follows that $P'Q'$ will also be a generator of the quadric, *i. e.* a generator of the quadric will remain thereon after the displacement.

It is a remarkable fact that, when the linear transformation is given, the infinite quadric is not definitely settled. We have seen how, in the first

place, the linear transformation must fulfil a fundamental condition; but, when that condition is obeyed, then a whole family of quadrics present themselves, any one of which is equally eligible for the infinite.

§ 26. **On the Intervene through which each Object is Conveyed.**—Given an object, x_1, x_2, x_3, x_4, find the intervene through which it is conveyed by the transformation, when

$$X_1X_2 + \lambda X_3X_4 = 0$$

is the infinite quadric.

In this equation I substitute for x_1, $x_1 + \theta y_1$, &c.; and, remembering that $Y_1 = \rho_1 X_1$, &c., we have,

$$(X_1 + \rho_1\theta X_1)(X_2 + \rho_2\theta X_2) + \lambda(X_3 + \rho_3\theta X_3)(X_4 + \rho_4\theta X_4) = 0,$$

or,

$$\begin{aligned} &\theta^2(\rho_1\rho_2X_1X_2 + \lambda\rho_3\rho_4X_3X_4) \\ &+ \theta\,(\rho_1X_1X_2 + \rho_2X_1X_2 + \lambda\rho_3X_3X_4 + \lambda\rho_4X_4X_4) \\ &+ X_1X_2 + \lambda X_3X_4 = 0. \end{aligned}$$

We simplify this by introducing

$$\rho_1\rho_2 = \rho_3\rho_4,$$

and writing $\lambda X_3X_4 \div X_1X_2 = \phi$, whence the equation becomes

$$\theta^2\rho_1\rho_2(1 + \phi) + \theta\,[(\rho_1 + \rho_2 + \phi(\rho_3 + \rho_4)] + (1 + \phi) = 0;$$

hence, if δ be the intervene, we have,

$$\cos\delta = \frac{1}{2\sqrt{\rho_1\rho_2}}\,\frac{\rho_1 + \rho_2 + \phi(\rho_3 + \rho_4)}{1 + \phi};$$

or, if we restore its value to ϕ,

$$\delta = \frac{1}{2\sqrt{\rho_1\rho_2}}\,\frac{X_1X_2(\rho_1 + \rho_2) + (\rho_3 + \rho_4)\,\lambda X_3X_4}{X_1X_2 + \lambda X_3X_4}.$$

COR. 1.—If
$$\rho_1 + \rho_2 = \rho_3 + \rho_4,$$
then
$$\cos\delta = \frac{\rho_1 + \rho_2}{2\sqrt{\rho_1\rho_2}};$$
i. e. all objects are translated through equal intervenes. This is the case which we shall subsequently have to consider under the title of the *vector*. In this case, as
$$\rho_1 + \rho_2 = \rho_3 + \rho_4,$$
and also,
$$\rho_1\rho_2 = \rho_3\rho_4,$$
we must have
$$\rho_1 = \rho_3, \text{ and } \rho_2 = \rho_4,$$
or
$$\rho_1 = \rho_4, \text{ and } \rho_2 = \rho_3.$$
In either case the equation for ρ will become a perfect square.

COR. 2.—Let
$$X_1 = 0,$$
then
$$\cos\delta = \frac{\rho_3 + \rho_4}{2\sqrt{\rho_3\rho_4}} \text{ (for } \rho_1\rho_2 = \rho_3\rho_4);$$
whence we find that all objects in the extent X_1 are displaced through equal intervenes. This intervene can be readily determined for
$$e^{i\delta} + e^{-i\delta} = \sqrt{\frac{\rho_3}{\rho_4}} + \sqrt{\frac{\rho_4}{\rho_3}},$$
whence
$$i\delta = \log\sqrt{\frac{\rho_3}{\rho_4}},$$
or
$$\delta = \frac{1}{2i}(\log\rho_3 - \log\rho_4).$$

The intervene through which every object on X_2 is conveyed has the same value.

We need not feel surprised that objects on X_1 and X_2 are all displaced through equal intervenes, for the locus of objects so displaced is a quadric of the form
$$X_1X_2 + HX_3X_4 = 0,$$

and, of course, for a special value of the distance this quadric becomes, simply

$$X_1X_2 = 0.$$

COR. 3.—If $X_1 = 0$, and $X_3 = 0$, then $\cos\delta$ becomes indeterminate; but this is as it should be, because all objects on X_1 and X_3 are at infinity.

COR. 4.—The pitch of the displacement can be simply determined, for an object on X_1X_2 is displaced through the intervene,

$$\frac{1}{2i}(\log\rho_3 - \log\rho_4),$$

and an object on X_3X_4, through the intervene,

$$\frac{1}{2i}(\log\rho_1 - \log\rho_2);$$

whence the pitch, or the ratio of the two intervenes is

$$\frac{\log\rho_3 - \log\rho_4}{\log\rho_1 - \log\rho_2}.$$

VIII.—APPLICATION OF THE ORTHOGONAL TRANSFORMATION.

§ 27. **The Orthogonal Transformation.**—This transformation is employed in Heath's Paper, already referred to (read at the Royal Society, on 17th January, 1884, *Phil. Trans.*, Part II., 1884). Of course, if I had known this at the time, I should have referred to the fact, when I subsequently used the same transformation (R. I. A., June 9, 1884, *Proceedings*, Ser. II., vol. iv., p. 252).

$$y_1 = (11)x_1 + (12)x_2 + (13)x_3 + (14)x_4,$$

$$y_2 = (21)x_1 + (22)x_2 + (23)x_3 + (24)x_4,$$

$$y_3 = (31)x_1 + (32)x_2 + (33)x_3 + (34)x_4,$$

$$y_4 = (41)x_1 + (42)x_2 + (43)x_3 + (44)x_4,$$

is the general type of transformation. It is said to be *orthogonal* if when x_1, &c., are solved in terms of y_1 we obtain as follows :—

$$x_1 = (11)y_1 + (21)y_2 + (31)y_3 + (41)y_4,$$
$$x_2 = (12)y_1 + (22)y_2 + (32)y_3 + (42)y_4,$$
$$x_3 = (13)y_1 + (23)y_2 + (33)y_3 + (43)y_4,$$
$$x_4 = (14)y_1 + (24)y_2 + (34)y_3 + (44)y_4.$$

From the first, the equation for ρ is, as before,

$$\begin{vmatrix} (11)-\rho & (12) & (13) & (14) \\ (21) & (22)-\rho & (23) & (24) \\ (31) & (32) & (33)-\rho & (34) \\ (41) & (42) & (43) & (44)-\rho \end{vmatrix} = 0.$$

From the second, the equation for ρ must be

$$\begin{vmatrix} (11)-\frac{1}{\rho} & (21) & (31) & (41) \\ (12) & (22)-\frac{1}{\rho} & (32) & (42) \\ (13) & (23) & (33)-\frac{1}{\rho} & (43) \\ (14) & (24) & (34) & (44)-\frac{1}{\rho} \end{vmatrix} = 0;$$

but we may interchange rows and columns in a determinant so that the last may be written,

$$\begin{vmatrix} (11)-\frac{1}{\rho} & (12) & (13) & (14) \\ (21) & (22)-\frac{1}{\rho} & (23) & (24) \\ (31) & (32) & (33)-\frac{1}{\rho} & (34) \\ (41) & (42) & (43) & (44)-\frac{1}{\rho} \end{vmatrix} = 0;$$

whence we see that the equation for ρ must be unaltered, if for ρ we substitute $\frac{1}{\rho}$. It must therefore be a reciprocal equation of the type,

$$\rho^4 + 4A\rho^3 + 6B\rho_2 + 4A\rho + 1 = 0,$$

and the roots are of the form

$$\alpha, \frac{1}{\alpha}, \quad \beta, \frac{1}{\beta};$$

and as

$$\alpha \frac{1}{\alpha} - \beta \frac{1}{\beta} = 0,$$

this transformation fulfils the fundamental condition. The orthogonal transformation provides, indeed, an obvious instance of a quadratic which the transformation fails to derange, for we have

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2.$$

§ 28. **On the Family of Quadrics which remain unaltered by the Homographic Transformation.**—It is also easily seen that the expression

$$x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = 0$$

is unchanged by the orthogonal transformation. We thus have the following quadric, which remains unaltered :—

$$x_1[(11)x_1+(12)x_2+(13)x_3+(14)x_4]+x_2[(21)x_1+(22)x_2+(23)x_3+(24)]x_4$$
$$+x_3[(31)x_1+(32)x_2+(33)x_3+(34)]x_4+x_4[(41)x_1+(42)x_2+(43)x_3+(44)]x_4=0;$$

or, writing it otherwise,

$$(11)x_1^2+(22)x_2^2+(33)x_3^2+(44)x_4^2+[(12)+(21)]\,x_1x_2+[(13)+(31)]\,x_1x_3$$
$$+[(14)+(41)]x_1x_4+[(23)+(32)]\,x_2x_3+[(24)+(42)]\,x_2x_4+[(43)+(34)]\,x_3x_4=0.$$

If this be denoted by U, and $x_1^2 + x_2^2 + x_3^2 + x_4^2$ by Ω, then, more generally, $U - h\Omega$ is unaltered by the transformation.

We now investigate the intervene θ, through which every object on

$$U - h\Omega = 0$$

is conveyed by the transformation.

If we substitute $x_1 + \lambda y_1$ in the equation $\Omega = 0$, we obtain

$$\Omega + 2\lambda U + \lambda^2\Omega,$$

and, accordingly, the intervene θ, through which an object is conveyed is defined by the equation,

$$\cos\theta = \frac{U}{\Omega};$$

hence the locus of objects moved through the intervene θ is simply

$$U - \Omega\cos\theta = 0.$$

§ 29. **Prove that U and Ω have Four Common Generators.**—The equation in ρ has four roots of the type

$$\rho', \frac{1}{\rho'}, \rho'', \frac{1}{\rho''}.$$

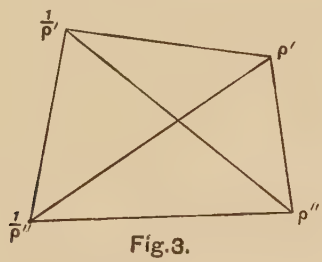


Fig. 3.

These correspond to the four vertices of the tetrahedron (fig. 3). The symmetry of the case shows that the two conjugate polars will be the ray joining

$$\frac{1}{\rho'} \text{ and } \rho',$$

and that joining

$$\frac{1}{\rho''} \text{ and } \rho''.$$

Let α_1, α_2, α_3, α_4, and β_1, β_2, β_3, β_4 be the co-ordinates of the corners at ρ' and ρ''. If, then, we substitute $\alpha_1 + \lambda\beta_1$, $\alpha_2 + \lambda\beta_2$ . . . &c., in Ω, it reduces to

$$2\lambda(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4).$$

Let us make the same substitution in U, we have, in general,

$$\begin{aligned} y_1 &= (11)x_1 + (12)x_2 + (13)x_3 + (14)x_4 \\ &= (11)(\alpha_1 + \lambda\beta_1) + (12)(\alpha_2 + \lambda\beta_2) +)13)(\alpha_3 + \lambda\beta_3) + (14)(\alpha_4 + \lambda\beta_4) \\ &= \rho\alpha_1 + \lambda\rho'\beta_1\,; \end{aligned}$$

whence, remembering that

$$U = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

$$U = (\alpha_1 + \lambda\beta_1)(\rho\alpha_1 + \lambda\rho'\beta_1) + (\alpha_2 + \lambda\beta_2)(\rho\alpha_2 + \lambda\rho'\beta_2) + \&c.,$$

and that α and β are both on Ω, we have,

$$U = \lambda(\rho + \rho')(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4)\,;$$

but since $\alpha\beta$ is a generator of Ω, the factor must vanish, and therefore $\alpha\beta$ is a generator of U.

It is thus proved that U has four generators in common with Ω.

§ 30. **The Vector.**—In the case of a vector, since, in general,

$$\cos\theta = \frac{U}{\Omega},$$

we have the following conditions :—

$$(11) = (22) = (33) = (44),$$

and also,

$$(12) + (21) = 0,$$

and the similar equations. In fact, U can only differ from Ω by a constant factor.

§ 31. **Verification of the Invariance of Intervene.**—As an exercise in the use of the orthogonal system of co-ordinates, I propose to demonstrate the following proposition :—

Let x_1, x_2, x_3, x_4, and x_1', x_2', x_3', x_4', be two objects which are conveyed by the transformation to y_1, y_2, y_3, y_4, and y_1', y_2', y_3', y_4', respectively, it is desired to show that the intervene between the two original points is equal to that between the transformed. The expressions for the cosine of the intervene are,

$$\frac{x_1x_1' + x_2x_2' + x_3x_3' + x_4x_4'}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}\,(x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2)^{\frac{1}{2}}},$$

and the similar one with y and y', instead of x and x'. The denominators are clearly equal, and we have only to notice that

$$x_1x_1' + x_2x_2' + x_3x_3' + x_4x_4' = y_1y_1' + y_2y_2' + y_3y_3' + y_4y_4'\,;$$

as an easy consequence of the formulæ connecting the orthogonal transformation.

§ 32. **Application of the Theory of Emanants.**—We can demonstrate the same proposition in a rather more instructive manner by reverting to the general case. The question affords an illustration of the theory of Emanants. (See Salmon, *Modern Higher Algebra.*)

Let $U = 0$ be a function of x_1, x_2, x_3, x_4. Let x_1', x_2', x_3', x_4' be a system of variables cogredient with x_1, x_2, x_3, x_4, and let us substitute in U the expressions $x_1 + kx_1'$, $x_2 + kx_2'$, &c., for x_1, x_2. The value of U then becomes

$$U + k\Delta U + \frac{k^2}{1.2}\Delta^2 U +, \text{ \&c.},$$

where,

$$\Delta = x_1'\frac{d}{dx_1} + x_2'\frac{d}{dx_2} + x_3'\frac{d}{dx_3} + x_4'\frac{d}{dx_4}.$$

If U be changed into V, a function of y, by the formulæ of transformation, we have, of course,

$$U = V;$$

but since y_1 is a linear function of x_1, &c., *i. e.*

$$y_1 = (11)x_1 + (12)x_2 + (13)x_3 + (14)x_4,$$

it follows that if we change x_1 into $x_1 + kx_1'$, &c., we simply change y_1 into $y_1 + ky_1'$. Hence we deduce, that if U be transformed by writing $x_1 + kx_1'$

&c., for x, then V will be similarly transformed by writing $y_1 + ky_1'$ for y, and, of course, as the original U and V were equal, so will the transformed U and V be equal. It further follows that as k is arbitrary, the several co-efficients will also be equal, and thus we have

$$x_1' \frac{dU}{dx_1} + \ldots = y_1' \frac{dU}{dy_1} + \ldots$$

Hence the intervene between two objects before displacement remains unaltered by that operation; for

$$\cos \delta = \frac{x \dfrac{dU}{dx_1} + \&c.}{\sqrt{U_x U_x'}} :$$

and by what we have just proved, this expression will remain unaltered if y be interchanged for x.

IX.—THE THEORY OF THE VECTOR.

§ 33. **The Vector in Orthogonal Co-ordinates.**—We have shown that the condition for a vector is expressed as follows:—

$$(11) = (22) = (33) = (44)$$

$$(12) + (21) = 0; \quad (13) + (31) = 0; \ \&c.$$

The orthogonal equations thus become—

$$+ (11) . (12) - (12) . (11) + (13) . (23) + (14) . (24) = 0,$$

$$+ (11) . (13) - (12) . (23) - (13) . (11) + (14) . (34) = 0,$$

$$+ (11) . (14) - (12) . (24) - (13) . (34) - (14) . (11) = 0,$$

$$+ (12) . (13) + (11) . (23) - (23) . (11) + (24) . (34) = 0,$$

$$+ (12) . (14) + (11) . (24) - (23) . (34) - (24) . (11) = 0,$$

$$+ (13) . (14) + (23) . (24) + (11) . (34) - (11) . (34) = 0,$$

$$+(11)^2+(12)^2+(13)^2+(14)^2=1,$$

$$+(12)^2+(11)^2+(23)^2+(24)^2=1,$$

$$+(13)^2+(23)^2+(11)^2+(34)^2=1,$$

$$+(14)^2+(24)^2+(34)^2+(11)^2=1.$$

We now introduce the following notation :—

$$(11)=\alpha;\quad (12)=\beta;\quad (13)=\gamma;\quad (14)=\delta,$$

and the equations reduce to

$$+\gamma(23)+\delta\ (24)=0, \qquad \text{(i)}$$

$$-\beta(23)+\delta\ (34)=0, \qquad \text{(ii)}$$

$$-\beta(24)-\gamma\ (34)=0, \qquad \text{(iii)}$$

$$+\beta\ \gamma\ +(24)(34)=0, \qquad \text{(iv)}$$

$$+\beta\ \delta\ -(23)(34)=0, \qquad \text{(v)}$$

$$+\gamma\ \delta\ +(23)(24)=0. \qquad \text{(vi)}$$

$$\left.\begin{aligned}
&+\alpha^2+\beta^2+\gamma^2+\delta^2=1,\\
&+\beta^2+\alpha^2+(23)^2+(24)^2=1,\\
&+\gamma^2+(23)^2+\alpha^2+(34)^2=1,\\
&+\delta^2+(24)^2+(34)^2+\alpha^2=1,
\end{aligned}\right\}\text{(vii)}$$

From (iv), $\beta\gamma=-(24)(34),$

From (v), $\beta\delta=+(23)(34);$

by multiplication,

$$\beta^2\gamma\delta=-(23)(24)(34)^2;$$

but, from (vi),

$$\gamma\delta=-(23)(24);$$

whence, we deduce,

$$\beta^2=(34)^2.$$

The significance of the double sign in the value of β will be afterwards apparent; for the present we take

$$(34) = +\beta.$$

From (ii) we get $\qquad (23) = +\delta,$

From (iii) we get $\qquad (24) = -\gamma,$

while the group (vii) will be satisfied if

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$

The scheme of orthogonal transformation for the *Right Vector* (for so we designate the case of $\beta = +34$), is as follows:—

$$\begin{vmatrix} +\alpha & +\beta & +\gamma & +\delta \\ -\beta & +\alpha & +\delta & -\gamma \\ -\gamma & -\delta & +\alpha & +\beta \\ -\delta & +\gamma & -\beta & +\alpha \end{vmatrix}.$$

If we append the condition

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1,$$

then we have the *Right Vector* completely defined.

We now take the other alternative,

$$\beta = -34;$$

then, from (ii), $\qquad \delta = -23,$

then, from (iii), $\qquad \gamma = +24.$

We thus have for the *Left Vector*, the form,

$$\begin{vmatrix} +\alpha & +\beta & +\gamma & +\delta \\ -\beta & +\alpha & -\delta & +\gamma \\ -\gamma & +\delta & +\alpha & -\beta \\ -\delta & -\gamma & +\beta & +\alpha \end{vmatrix},$$

with, as before, the condition,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$

These forms of right and left vectors are fundamental in the theory. The conception of the vector transformation of the content is due to Clifford.

§ 34. **The Intervene corresponding to a Vector.**—Let the object x_1, x_2, x_3, x_4, be conveyed by a right vector to y_1, y_2, y_3, y_4. Then,

$$y_1 = +\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4,$$

$$y_2 = -\beta x_1 + \alpha x_2 + \delta x_3 - \gamma x_4,$$

$$y_3 = -\gamma x_1 - \delta x_2 + \alpha x_3 + \beta x_4,$$

$$y_4 = -\delta x_1 + \gamma x_2 - \beta x_3 + \alpha x_4.$$

If θ be the intervene, we have,

$$\cos\theta = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 = \alpha.$$

Hence we see that the *leading term of a vector is the cosine of the intervene through which the vector displaces every object of the content.* The same demonstration applies equally to the left vector.

We also find that

$$\sin^2\theta = \beta^2 + \gamma^2 + \delta^2.$$

For, of course,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$

§ 35. **Parallel Vectors.**—A system of vectors which have the same relative values of β, γ, δ, are termed *Parallel.*

We now prove the following important theorem due to Clifford. A system of right vectors, which are respectively parallel, intersect two generators of one system, while a parallel system of left vectors intersect two generators of the other system.

A generator intersected by two ranges from a right vector will be defined by the points,

$$+\alpha, \quad -\beta, \quad -\gamma, \quad -\delta,$$

$$+\beta, \quad +\alpha', \quad -\delta, \quad +\gamma,$$

while a generator intersected by two ranges from a left vector will be defined by

$$\begin{matrix} +\alpha_0', & -\beta_0, & -\gamma_0, & -\delta_0, \\ +\beta_0, & +\alpha_0', & +\delta_0, & -\gamma_0. \end{matrix}$$

To prove the theorem, it is only necessary to show that these four points are coplanar, for then the two generators intersect, *i. e.* are of opposite systems. We have, then, only to show that the following determinant vanishes :—

$$\begin{vmatrix} \alpha' & -\beta & -\gamma & -\delta \\ \beta & \alpha' & -\delta & \gamma \\ \alpha'_0 & -\beta_0 & -\gamma_0 & -\delta_0 \\ \beta_0 & \alpha_0' & \delta_0 & -\gamma_0 \end{vmatrix}.$$

This will be most readily shown by squaring, for it then reduces to the simple form,

$$\begin{vmatrix} 0 & 0 & [\alpha'\alpha_0'] & [\alpha'\beta_0] \\ 0 & 0 & [\beta\alpha_0'] & [\beta\beta_0] \\ [\alpha\alpha_0'] & [\beta\alpha_0'] & 0 & 0 \\ [\beta_0\alpha'] & [\beta\beta_0] & 0 & 0 \end{vmatrix},$$

whence we see that the original determinant is simply,

$$\begin{vmatrix} [\alpha'\alpha_0'] & [\beta\alpha_0'] \\ [\beta_0\alpha'] & [\beta\beta_0] \end{vmatrix};$$

which expanded, becomes,

$$\begin{aligned} &(\alpha'\alpha_0' + \beta\beta_0 + \gamma\gamma_0 + \delta\delta_0)(\beta\beta_0 + \alpha'\alpha_0' - \delta\delta_0 - \gamma\gamma_0), \\ &-(\alpha'\beta_0 - \beta\alpha_0' - \gamma\delta_0 + \delta\gamma_0)(\beta\alpha_0' - \alpha'\beta_0 + \delta\gamma_0 - \gamma\delta_0), \\ &= (\alpha'\alpha_0' + \beta\beta_0)^2 - (\gamma\gamma_0 + \delta\delta_0)^2 + (\alpha'\beta_0 - \alpha_0'\beta)^2 - (\delta\gamma_0 - \gamma\delta_0)^2, \\ &= \alpha'^2\alpha_0'^2 + \beta^2\beta_0^2 + \alpha'^2\beta_0^2 + \alpha_0'^2\beta^2 - \gamma^2\gamma_0^2 - \delta^2\delta_0^2 - \delta^2\gamma_0^2 - \gamma^2\delta_0^2, \\ &= \alpha'^2(\alpha_0'^2 + \beta_0^2) + \beta^2(\alpha_0'^2 + \beta_0^2) - \gamma^2(\gamma_0^2 + \delta_0^2) - \delta^2(\gamma_0^2 + \delta_0^2), \\ &= (\alpha'^2 + \beta^2)(\alpha_0'^2 + \beta_0^2) - (\gamma^2 + \delta^2)(\gamma_0^2 + \delta_0^2); \end{aligned}$$

but, $$\alpha'^2 + \beta^2 + \gamma^2 + \delta^2 = 0;$$

whence this expression is

$$(\alpha'^2 + \beta^2)(\alpha_0'^2 + \beta_0^2 + \gamma_0^2 + \delta_0^2) = 0.$$

This reduction will appear the more remarkable if we calculate the determinant on the supposition that the vectors were homonymous. It is,

$$\begin{vmatrix} \alpha' & -\beta & -\gamma & -\delta \\ \beta & \alpha' & -\delta & \gamma \\ \alpha_0' & -\beta_0 & -\gamma_0 & -\delta_0 \\ \beta_0 & \alpha_0' & -\delta_0 & \gamma_0 \end{vmatrix}.$$

Squaring, we get, as before,

$$\begin{vmatrix} [\alpha'\alpha_0'] & [\beta\alpha_0'] \\ [\beta_0\alpha'] & [\beta\beta_0] \end{vmatrix};$$

but now,

$$[\alpha'\alpha_0'] = [\beta\beta_0],$$

$$[\beta_0\alpha'] = -[\beta\alpha_0'];$$

whence the determinant reduces to

$$[\alpha'\alpha_0']^2 + [\beta\alpha_0']^2,$$

a value very different to that in the former case.

§ 36. **The Co-ordinates of the Vector.**—We now see, that while the intervene of the vector is defined by α, from the equation,

$$\cos\theta = \alpha,$$

yet that the *direction* of the vector is given by β, γ, δ, which three quantities indicate the two generators of the infinite, which the vector intersects.

§ 37. **The Composition of Vectors.**—Let an object x be conveyed to y by the operation of a vector, and let the object y be then conveyed to z by

the operation of a second vector, which we shall first suppose to be homonymous (*i. e.* both right or both left) with the preceding. Then we have, from the first,

$$y_1 = +\,\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4,$$

$$y_2 = -\,\beta x_1 + \alpha x_2 + \delta x_3 - \gamma x_4,$$

$$y_3 = -\,\gamma x_1 - \delta x_2 + \alpha x_3 + \beta x_4,$$

$$y_4 = -\,\delta x_1 + \gamma x_2 - \beta x_3 + \alpha x_4,$$

and, from the second vector,

$$z_1 = +\,\alpha' y_1 + \beta' y_2 + \gamma' y_3 + \delta' y_4,$$

$$z_2 = -\,\beta' y_1 + \alpha' y_2 + \delta' y_3 - \gamma' y_4,$$

$$z_3 = -\,\gamma' y_1 - \delta' y_2 + \alpha' y_3 + \beta' y_4,$$

$$z_4 = -\,\delta' y_1 + \gamma' y_2 - \beta' y_3 + \alpha' y_4,$$

Substituting for y_1, y_2, y_3, y_4, we obtain the following values for z_1, z_2, z_3, z_4.

The right vector, $\alpha, \beta, \gamma, \delta$, followed by the right vector, $\alpha', \beta', \gamma', \delta'$—

$$z_1 = \overline{+\,\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\,x_1 \quad \overline{+\,\alpha\beta' + \beta\alpha' + \gamma\delta' - \delta\gamma'}\,x_2 \quad \overline{\alpha\gamma' - \beta\delta' + \gamma\alpha' + \delta\beta'}\,x_3 \quad \overline{+\,\alpha\delta' + \beta\gamma' - \gamma\beta' + \delta\alpha'}\,x_4,$$

$$z_2 = \overline{-\,\alpha\beta' - \beta\alpha' - \gamma\delta' + \delta\gamma'}\,x_1 \quad \overline{+\,\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\,x_2 \quad \overline{\alpha\delta' + \beta\gamma' - \gamma\beta' + \delta\alpha'}\,x_3 \quad \overline{-\,\alpha\gamma' + \beta\delta' - \gamma\alpha' - \delta\beta'}\,x_4,$$

$$z_3 = \overline{-\,\alpha\gamma' + \beta\delta' - \gamma\alpha' - \delta\beta'}\,x_1 \quad \overline{-\,\alpha\delta' - \beta\gamma' + \gamma\beta' - \delta\alpha'}\,x_2 \quad \overline{\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\,x_3 \quad \overline{+\,\alpha\beta' + \beta\alpha' + \gamma\delta' - \delta\gamma'}\,x_4,$$

$$z_4 = \overline{-\,\alpha\delta' - \beta\gamma' + \gamma\beta' - \delta\alpha'}\,x_1 \quad \overline{+\,\alpha\gamma' - \beta\delta' + \gamma\alpha' + \delta\beta'}\,x_2 \quad \overline{-\,\alpha\beta' - \beta\alpha' - \gamma\delta' + \delta\gamma'}\,x_3 \quad \overline{+\,\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\,x_4.$$

The right vector, $\alpha', \beta', \gamma', \delta'$, followed by the right vector, $\alpha, \beta, \gamma, \delta$—

$$z_1 = \overline{+\,\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\,x_1 \quad \overline{+\,\alpha\beta' + \beta\alpha' - \gamma\delta' + \delta\gamma'}\,x_2 \quad \overline{+\,\alpha\gamma' + \beta\delta' + \gamma\alpha' - \delta\beta'}\,x_3 \quad \overline{+\,\alpha\delta' - \beta\gamma' + \gamma\beta' + \delta\alpha'}\,x_4,$$

$$z_2 = \overline{-\,\alpha\beta' - \beta\alpha' + \gamma\delta' - \delta\gamma'}\,x_1 \quad \overline{+\,\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\,x_2 \quad \overline{+\,\alpha\delta' - \beta\gamma' + \gamma\beta' + \delta\alpha'}\,x_3 \quad \overline{-\,\alpha\gamma' - \beta\delta' - \gamma\alpha' + \delta\beta'}\,x_4,$$

$$z_3 = \overline{-\,\alpha\gamma' - \delta\beta' - \gamma\alpha' + \delta\beta'}\,x_1 \quad \overline{-\,\alpha\delta' + \beta\gamma' - \gamma\beta' - \delta\alpha'}\,x_2 \quad \overline{+\,\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\,x_3 \quad \overline{+\,\alpha\beta' + \beta\alpha' - \gamma\delta' + \delta\gamma'}\,x_4,$$

$$z_4 = \overline{-\,\alpha\delta' + \beta\gamma' - \gamma\beta' - \delta\alpha'}\,x_1 \quad \overline{+\,\alpha\gamma' + \beta\delta' + \gamma\alpha' - \delta\beta'}\,x_2 \quad \overline{-\,\alpha\beta' - \beta\alpha' + \gamma\delta' - \delta\gamma'}\,x_3 \quad \overline{+\,\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\,x_4.$$

We thus learn the important truth, that when two or more *homonymous* vectors are compounded, the order of their application must be carefully specified. For example, if the object x be first transposed by the vector α and then by α', it attains a different position to that it would have gained if first transposed by α' and then by α.

We see, however, that in either case two homonymous vectors compound into a vector homonymous with the two components.

We shall now study the case of the composition of heteronymous vectors.

The right vector, α, β, γ, δ, followed by the left vector, α', β', γ', δ'—

$$z_1 = \overline{+\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\, x_1 \quad \overline{+\alpha\beta' + \beta\alpha' + \gamma\delta' - \delta\gamma'}\, x_2 \quad \overline{+\alpha\gamma' - \beta\delta' + \gamma\alpha' + \delta\beta'}\, x_3 \quad \overline{+\alpha\delta' + \beta\gamma' - \gamma\beta' + \delta\alpha'}\, x_4,$$

$$z_2 = \overline{-\alpha\beta' - \beta\alpha' + \gamma\delta' - \delta\gamma'}\, x_1 \quad \overline{+\alpha\alpha' - \beta\beta' + \gamma\gamma' + \delta\delta'}\, x_2 \quad \overline{-\alpha\delta' - \beta\gamma' - \gamma\beta' + \delta\alpha'}\, x_3 \quad \overline{+\alpha\gamma' - \beta\delta' - \gamma\alpha' - \delta\beta'}\, x_4,$$

$$z_3 = \overline{-\alpha\gamma' - \beta\delta' - \gamma\alpha' + \delta\beta'}\, x_1 \quad \overline{+\alpha\delta' - \beta\delta' - \gamma\beta' - \delta\alpha'}\, x_2 \quad \overline{+\alpha\alpha' + \beta\beta' - \gamma\gamma' + \delta\delta'}\, x_3 \quad \overline{-\alpha\beta' + \beta\alpha' - \gamma\delta' - \delta\gamma'}\, x_4,$$

$$z_4 = \overline{-\alpha\delta' + \beta\gamma' - \gamma\beta' - \delta\alpha'}\, x_1 \quad \overline{-\alpha\gamma' - \beta\delta' + \gamma\alpha' - \delta\beta'}\, x_2 \quad \overline{+\alpha\beta' - \beta\alpha' - \gamma\delta' - \delta\gamma'}\, x_3 \quad \overline{+\alpha\alpha' + \beta\beta' + \gamma\gamma' - \delta\delta'}\, x.$$

The left vector, α', β', γ', δ', followed by the right vector, α, β, γ, δ—

$$z_1 = \overline{+\alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta'}\, x_1 \quad \overline{+\alpha\beta' + \beta\alpha' + \gamma\delta' - \delta\gamma'}\, x_2 \quad \overline{+\alpha\gamma' - \beta\delta' + \gamma\alpha' + \delta\beta'}\, x_3 \quad \overline{+\alpha\delta' + \beta\gamma - \gamma\beta' + \delta\alpha'}\, x_4,$$

$$z_2 = \overline{-\alpha\beta' - \beta\alpha' + \gamma\delta' - \delta\gamma'}\, x_1 \quad \overline{+\alpha\alpha' - \beta\beta' + \gamma\gamma' + \delta\delta'}\, x_2 \quad \overline{-\alpha\delta' - \beta\gamma' - \gamma\beta' + \delta\alpha'}\, x_3 \quad \overline{+\alpha\gamma' - \beta\delta' - \gamma\alpha' - \delta\beta'}\, x_4,$$

$$z_3 = \overline{-\alpha\gamma' - \beta\delta' - \gamma\alpha' + \delta\beta'}\, x_1 \quad \overline{+\alpha\delta' - \beta\gamma' - \gamma\beta' - \delta\alpha'}\, x_2 \quad \overline{+\alpha\alpha' + \beta\beta' - \gamma\gamma' + \delta\delta'}\, x_3 \quad \overline{-\alpha\beta' + \beta\alpha' - \gamma\delta' - \delta\gamma'}\, x_4,$$

$$z_4 = \overline{-\alpha\delta' + \beta\gamma' - \gamma\beta' - \delta\alpha'}\, x_1 \quad \overline{-\alpha\gamma' - \beta\delta' + \gamma\alpha' - \delta\beta'}\, x_2 \quad \overline{+\alpha\beta' - \beta\alpha' - \gamma\delta' - \delta\gamma'}\, x_3 \quad \overline{+\alpha\alpha' + \beta\beta' + \gamma\gamma - \delta\delta'}\, x_4.$$

We thus learn the remarkable fact, that if a right vector be followed by a left vector, the effect produced is the same as if the order of the two vectors were interchanged.

This is *not true* for two right vectors or two left vectors.

The theorems at which we have arrived may be thus generally enunciated:—

In the composition of vectors the order of two heteronymous vectors does not affect the result, but that of two homonymous vectors does affect the result.

In the composition of two homonymous vectors the result is also an homonymous vector. In the composition of two heteronymous vectors the result is not a vector at all.

§ 38. **Another Investigation.**—The law of permutability of heteronymous vectors can be demonstrated by geometry.

Let AB and $A'B'$ be a pair of right generators, and CD and $C'D'$ a pair

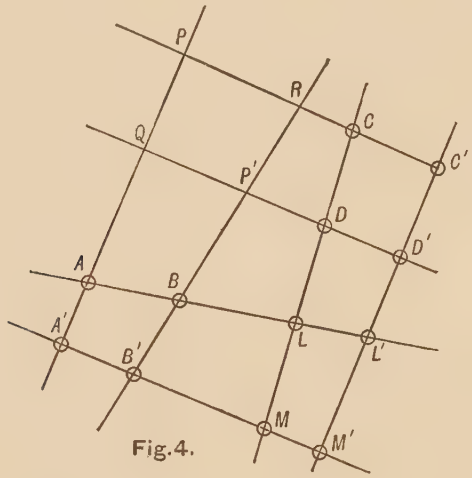


Fig.4.

of left generators. Let the right vector convey P to Q, and then let the left vector carry Q to the final position P'. We shall now show that P' would have been equally reached if P had gone first to R, so that $PR = QP'$, and that then R was conveyed by the vector, $RB'B$, through a distance equal to PQ.

Draw through P the transversal $PRCC'$. Take R, so that

$$(PRCC') = (QP'DD');$$

but, because this relation holds,

$$PQ, \quad RP', \quad CD, \quad C'D'$$

must be cohyperboloidal.

Therefore RP' must intersect AL and $A'M$, and therefore, also,

$$(PQAA') = (RP'BB').$$

Hence, finally,

$$PQ = RP' \text{ and } PR = QP'.$$

§ 39. **Determination of the Two Heteronymous Vectors equivalent to a given Motor.**—If a right vector, $\alpha, \beta, \gamma, \delta$, be followed by a left vector, $\alpha', \beta', \gamma', \delta'$, then the result obtained is a displacement of the most general type called a motor. I now show how a right vector and a left vector can be uniquely determined so as to accomplish any required homographic transformation that conserves intervene.

For, if we identify the several co-efficients, we obtain equations of the type,

$$(11) = \alpha\alpha' - \beta\beta' - \gamma\gamma' - \delta\delta',$$

$$(21) = -\alpha\beta' - \beta\alpha' + \gamma\delta' - \delta\gamma',$$

$$(31) = -\alpha\gamma' - \beta\delta' - \gamma\alpha' + \delta\beta',$$

$$(41) = -\alpha\delta' + \beta\gamma' - \gamma\beta' - \delta\alpha'.$$

These can be simply reduced to a linear form; for multiply the first by α', and the second, third, and fourth by $-\beta', -\gamma', -\delta'$, respectively, and add, we obtain,

$$(11)\alpha' - (21)\beta' - (31)\gamma' - (41)\delta' = \alpha;$$

for,

$$\alpha'^2 + \beta'^2 + \gamma'^2 + \delta'^2 = 1.$$

In a similar manner we obtain a number of analogous equations, which are here all brought together for convenience—

$$\begin{aligned}
(11)\alpha' - (21)\beta' - (31)\gamma' - (41)\delta' &= \alpha,\\
-(21)\alpha' - (11)\beta' + (41)\gamma' - (31)\delta' &= \beta,\\
-(31)\alpha' - (41)\beta' - (11)\gamma' + (21)\delta' &= \gamma,\\
-(41)\alpha' + (31)\beta' - (21)\gamma' - (11)\delta' &= \delta.\\
+(22)\alpha' + (12)\beta' - (42)\gamma' + (32)\delta' &= \alpha,\\
+(12)\alpha' - (22)\beta' - (32)\gamma' - (42)\delta' &= \beta,\\
+(42)\alpha' - (32)\beta' + (22)\gamma' + (12)\delta' &= \gamma,\\
-(32)\alpha' - (42)\beta' - (12)\gamma' + (22)\delta' &= \delta.
\end{aligned}$$

$$+(33)\alpha' + (43)\beta' + (13)\gamma' - (23)\delta' = \alpha,$$

$$+(43)\alpha' + (33)\beta' - (23)\gamma' - (13)\delta' = \beta,$$

$$-(13)\alpha' - (23)\beta' - (33)\gamma' - (43)\delta' = \gamma,$$

$$+(23)\alpha' + (13)\beta' - (43)\gamma' + (33)\delta' = \delta.$$

$$+(44)\alpha' - (34)\beta' + (24)\gamma' + (14)\delta' = \alpha,$$

$$+(34)\alpha' + (44)\beta' + (14)\gamma' - (24)\delta' = \beta,$$

$$-(24)\alpha' - (14)\beta' + (44)\gamma' - (34)\delta' = \gamma,$$

$$+(14)\alpha' - (24)\beta' - (34)\gamma' - (44)\delta' = \delta.$$

These will enable α, β, γ, δ, and α', β', γ', δ' to be determined.

§ 40. **A Concise Expression for the Vector.**—It will often be convenient to express the vector in the form,

$$\begin{vmatrix} \cos\theta, & \alpha\sin\theta, & \beta\sin\theta, & \gamma\sin\theta \\ -\alpha\sin\theta, & \cos\theta, & \gamma\sin\theta, & -\beta\sin\theta \\ -\beta\sin\theta, & -\gamma\sin\theta, & \cos\theta, & \alpha\sin\theta \\ -\gamma\sin\theta, & \beta\sin\theta, & -\alpha\sin\theta, & \cos\theta \end{vmatrix},$$

in which case the intervene is θ, and its direction cosines are α, β, γ, subject to the condition,

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

§ 41. **The Departure between Two Vectors.**—This is now very easily found.

Let the object, 1, 0, 0, 0, be displaced by the first vector, and it will come to

$$\cos\theta, \quad -\alpha\sin\theta, \quad -\beta\sin\theta, \quad -\gamma\sin\theta'.$$

If it had been displaced by the second vector from its original position, it would have attained,

$$\cos\theta', \quad -\alpha'\sin\theta', \quad -\beta'\sin\theta', \quad -\gamma'\sin\theta.$$

Now if ϕ be the intervene between these points, then

$$\cos\phi = \cos\theta\cos\theta' + \sin\theta\sin\theta'(\alpha\alpha' + \beta\beta' + \gamma\gamma').$$

But, from Heath's theorem, we see that if ω be the departure between the vectors,

$$\cos\phi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\omega;$$

whence the remarkable result,

$$\cos\omega = \alpha\alpha' + \beta\beta' + \gamma\gamma'.$$

This theorem is so important that I shall give it the more general investigation by taking, as the initial object, any x_1, x_2, x_3, x_4, instead of the special object, 1, 0, 0, 0. We have then, for $\cos\phi$, the value,

$$\begin{aligned}
&+(\cos\theta x_1 + \alpha\sin\theta x_2 + \beta\sin\theta x_3 + \gamma\sin\theta x_4)(\cos\theta' x_1 + \alpha'\sin\theta' x_2 + \beta'\sin\theta' x_3 + \gamma'\sin\theta' x_4),\\
&+(-\alpha\sin\theta x_1 + \cos\theta x_2 + \gamma\sin\theta x_3 - \beta\sin\theta x_4)(-\alpha'\sin\theta' x_1 + \cos\theta' x_2 + \gamma'\sin\theta' x_3 - \beta'\sin\theta' x_4),\\
&+(-\beta\sin\theta x_1 - \gamma\sin\theta x_2 + \cos\theta x_3 + \alpha\sin\theta x_4)(-\beta'\sin\theta' x_1 - \gamma'\sin\theta' x_2 + \cos\theta' x_3 + \alpha'\sin\theta' x_4)\\
&+(-\gamma\sin\theta x_1 + \beta\sin\theta x_2 - \alpha\sin\theta x_3 + \cos\theta x_4)(-\gamma'\sin\theta' x_1 + \beta'\sin\theta' x_2 - \alpha'\sin\theta' x_3 + \cos\theta' x_4),
\end{aligned}$$

which reduces, as before, to

$$\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\omega.$$

If, therefore, ω be the departure between two vectors, we have, as before,

$$\cos\omega = \alpha\alpha' + \beta\beta' + \gamma\gamma'.$$

CONCLUSION.

In a previous communication to this Academy I demonstrated that a force on a vector was neutral with regard to a displacement on a heteronymous vector. Of cognate significance in the theory are the principles now laid down: that, the order of two heteronymous vectors is convertible, without affecting the resultant, which is not a vector at all, and that, though two homonymous vectors always compound into an homonymous vector, the order of their application is not convertible without producing quite a different resultant.

It appears to me that these principles will be of fundamental importance when the generalized conception of Kinematics and Dynamics shall be thoroughly investigated.

VI.

THE TRIPARTITE LIFE OF ST. PATRICK: NEW TEXTUAL STUDIES.

By The Rev. B. MAC CARTHY, D.D., M.R.I.A., &c.

[Read January 14, 1889.]

That upon the recurrence of his festival a sketch of the life and labours of St. Patrick should be delivered in the churches of Ireland would be a procedure in mere conformity with ecclesiastical usage. Here, as on the Continent, Saints' days were celebrated. Preaching formed, on such occasions, an integral item in the ceremonial.* Reverence for the national Apostle did not, however, confine itself within the limits of customary observances. The "solemnity of his falling asleep," we learn upon reliable authority, was to be honoured throughout three days and three nights by relaxation of the Lenten rigour.† The remission, it is safe to infer, was an adjunct of the anniversary festivity. That a panegyric was delivered to the people thrice during the Triduum is proved by the fact, that a discourse of the kind, distributed into three divisions, has been preserved. Each section is prefaced with an appropriate scriptural text, and concludes with the declaration that the miracles are to be carried so far upon that day.

* "Tri herdaige dlegar do denum i sollamnaib na noem ocus na fhiren—ise in cetna erddach, celebrad ocus procept brethre De, &c.—Three festivities it is appointed to perform on solemnities of the saints and of the just. The first festivity is celebration (of the office) and preaching of the word of God."—*Lebar Brec*, p. 73 *a*: *cf.* p. 169 *a*.

† "Sollemptatate[m] dormitionis ejus honorari in medio veris per tres dies et tres noctes omni bono cibo præter carnem." Book of Armagh, fol. 16 *a*. *Per* is placed above *iii* (*tres*). The scribe overlooked it at first. Compare *tri la ocus teora aidchi do icon prociupt*—three days and three nights for him (was he) at the preaching.—Tripartite, Part III. Eg.

When the acts of the Saint began to be digested from oral * (along with other) sources, the materials here collected, having the warranty of periodical publicity, were doubtless amongst the first to be employed. In time, the sermons were themselves reduced to literary form. The consequence followed, as was to be expected. An alteration, not indeed in form, but none the less radical, took effect in the original design. In truth, a work of the character led almost of necessity to interpolation. Scholiasts and scribes, ignorant or heedless of what they worked upon, made accumulated additions. The accretions varied in language, in subject, and in purport. Thereby gradually the primitive features almost disappeared, overladen with a mass of matter widely at variance with the spirit of Hagiography.† Hence, likewise, the MS. rubric—Life of Patrick; which, together with the structure, most probably led Colgan to the adoption of the not inappropriate title, *Vita Tripartita.*‡ In its present form, accordingly, the Tripartite, like portions of the Book of Armagh, is rather a plea for the privileges of the primatial See than a eulogy of the Apostle of Ireland.

Due regard being had to the circumstances, the non-contentious portions, it can thus be concluded, may be assigned to a remote antiquity.

With reference to the language, the change has been of a character scarcely less fundamental. The text, as far as our present information extends, exists, not quite in its entirety, only in two MSS. One is preserved in the British Museum, Egerton, 93; the other in the Bodleian Library, Oxford, Rawlinson, B 512. That they were derived mediately from a common archetype, published data are sufficiently reliable to

* "Haec de multis Sancti Patricii gestis . . . memoria labili . . . explicare aggrediar." Book of Armagh, fol. 20 *a*. "Tirechan, episcopus, haec scripsit ex ore vel libro Ultani," &c.—Fol. 9 *b*.

† In one place, for instance, a Columban monastery is openly taxed with dishonourable conduct. Maigen inna [fh]arrad andess la Patraic. Fer muinteri do conaggaib, mac Diciull (Dicuill). La Colum Cille indiu tre foill—A place in its vicinity to the south (belongs) to Patrick. A man of his community, Mac Dicuill, founded it. It belongs to Colum Cille to-day through treachery. Tripartite, Part I., R., fol. 10 *a*. The corresponding folio has been removed from the Egerton MS.

‡ *Triadis Thaumaturgae: seu divorum Patricii, Columbae et Brigidae . . . acta.* Lovanii, 1647. Septima Vita Tripartita S. Patricii. p. 117.

decide.* Another conclusion a cursory examination places beyond controversy. Both, it is matter of keen regret, have been executed by scribes whose ignorance of the Latin was perhaps equalled by their unacquaintance with the older phases of the native speech. Two examples will suffice to illustrate their unfitness for the work of transcription.

The following passage, partly Irish and partly Latin, occurs in the Second Part:—"Asbert (Patraic) ropad chongbail do ocus cathir docum septem episcopis." *Doc(h)um* is an Irish preposition signifying *towards*. It is employed after verbs of motion, and governs the genitive. Here, consequently, it is meaningless, and must be a corruption. The form is to be resolved into *do* and *cum*. The translation is thus:—"(Patrick) said that it (Racoon, Co. Donegal) would be a habitation for him and an establishment *for him*, *with* seven bishops." The scribes, namely, joined on the Latin *cum* to the Irish *do* (*to him*).

At the end of the Third Part, a title is found as follows:—

Na cerdae oc denum nammias ocus na menistrech ocus na cailech n-altora.

Menistrech is genitive plural of *menister* (syncopated form, *menster*), a loan-word from the Latin *ministerium*. The word will be found in Du-Cange. Of the references there given in support of the meaning assigned, that to Gregory of Tours is singularly apposite in the present instance:—

Dum Ararim fluvium [Saône] transire cuperet, nave impleta demergitur, ac ipse sacerdos pelago operitur, habens ad collum cum Evangeliorum libro ministerium quotidianum, id est, patenulam parvam cum calice.†

The sense being thus determined, the text can be restored with facility. "Na cerdae oc denum na menistrech; idon, nammias ocus na cailech n-altora—the artizans a-making the service-sets; namely, the

* Note A.

† Greg. Tur., *De gloria Confessorum*, xxii. (De Maximo Abbate). *Bibliotheca Maxima Patrum*, Lugduni, 1677, tom. 11, p. 876, G. In a canon of the Hubertine Penitential, hitherto unnoticed in this connexion, *ministerium* is taken in a more comprehensive sense. Si quis de ministerio ecclesiæ fraudaverit, id est, vasa altaris, pallas, vel sindonem subtractum vendiderit, Judæ similis æstimabitur (Pœn. Hubert., can. xliii.).

patens and the chalices of the altar." The words: *idon, nammias ocus na cailech n-altora* at first formed a marginal gloss upon *menistrech.* Like so many of the class in our MSS., they subsequently became incorporated in the text. But the wrong position, selected evidently at hap-hazard, in which we find them, puts it beyond question that the insertion was effected by scribes to whom the meaning was an enigma.

To attempt to determine the date of composition from the historical references and grammatical forms of the known extant codices were, therefore, utterly to misconceive the history and scope of the Tripartite.

Emendation of the text and discrimination of the contents are, consequently, of primary necessity to produce results upon which reliance can be placed. A beginning in both directions the Academy's photographed issue of the Book of Ballymote enables us to effect. Of all places, perhaps the least promising in linguistic aid, outside declensional forms, is a series of genealogies. Notwithstanding, right in the midst of the pedigrees of Muscraidhe Tire (a territory comprising the baronies of Upper and Lower Ormond, Co. Tipperary), it was my good fortune to light upon portion of the Tripartite.* Its presence was caused in this way. Having occasion to mention the three sons of Forat, son of Conla, the compiler, or (more probably) a scholiast, evidently a tribesman or dependent, availed himself of the opportunity to introduce their connexion with St. Patrick, as he found it set forth in the Tripartite.† The passages consist of one in Irish and another in Latin. The first, luckily, is in verse. Though in part corrupt, yet by its assistance the original form of the quatrains can be restored.

The opening stanza reads as follows in Egerton.‡ (The MS. lines run on continuously):—

I.	I.
Creitis Muinnech mar	Muinnech the great believes
Do Patraic ría cach;	In Patrick before everyone;
Combia for a thuaith	So that there will be over his territory
Toisech uad co brath.	A chief from him to doom.

* The Tripartite excerpts, contained elsewhere in the Book of Ballymote, will form the subject of another Paper.

† Note B. ‡ Note C.

The metre which this is composed in is called *Dechnad bec*, or minor Dechnad, in the Tract on Versification in the Book of Ballymote. Each stanza, we are informed, consists of four *deviations*, or lines.* These, it is explained, are equivalent to twenty *joints*, or syllables. The line, accordingly, contained five syllables, the final of which was to form a word. The foregoing, consequently, is correct in scansion and structure. The variants of B (Book of Ballymote) are :—

Robe fo rath uaith	There will be under singular felicity
Rigi uad cobrath.	Kingship from him to doom.

But, to judge from E (Egerton) and R (Rawlinson), *fo rath uaith* should be distributed differently—*for a thuaith.* The change harmonizes the reading of B with the lection of the two other MSS.

Verse 2 runs thus in E :—

II.	II.
Creitis Mechar cer[p],†	Mechar the vigorous believes,
Pa fer condilc fir;	He was a man of true judgment ;
Dobert Patraic bendacht mbuain—	Patrick gave a lasting blessing—
Cetlad do fri rig.	Companionship for him with a king.

The variants of B are :—

Ba fer con deig fir,	He was a man with (= of) good truth,
Dobert bennacht mbuain—	He gave a lasting blessing—

Here *fir* must be corrupt; unless, which may not be unlikely, the grammar, as in other places, was sacrificed to the necessity of the verse. The correct case would be *fire*, the vocable being an *ia*-stem. But, as we have seen, the line must terminate in a monosyllable.

In the third line B is clearly correct. *Patraic* was added in the

* P. 297 *b*, ll. 5–17 : *cf.* p. 302 *b*, ll. 24–6.

† The MS. gives : *cerpa fer.* This signifies : *Though he was a man*, &c. The scribe, namely, altered the original to suit his own meaning ; to the ruin of the sense and metre.

margin of E and R, to show that it was the subject of *Dobert.* Subsequently the gloss became inserted in the text.

The final stanza is given as follows in E:—

Frithmbert in fer fercach, Fuirgg,	The angry man, Fuirgg, opposed,
Ciar bu rigla[e]ch liath ;	Though he was a royal hero, hoary ;
Adal fu diud iar cach,	His assembly at the end after every one,
Bith a min cobrath, ni liach.	To be in misery to doom, (is) not a pity.

The opening line runs the same as the foregoing in R. and B. But it manifestly exceeds the metric limit by no less than two syllables. The correction can be made by the omission of *in fer:* which leaves an idiomatic expression, *Frithmbert fercach*—" he opposed angry " = angrily. The complement of the predicate, namely, is expressed by the adjective qualifying the subject, in place of the adverb qualifying the verb. Compare petam connubia supplex of the *Æneid*, and the line in *Lycidas:*

He shook his mitred locks, and stern bespake.

In line 2, the true reading, *riglaech*, not *riglach* as in E and R, has been preserved in B.

Line 3 is very depraved in E and R. As it stands, emendation seems almost impossible. No doubt, the omission of *iar* and placing *cach* in the genitive (*caich*) upon *diud* would be grammatically tenable. But the expression " *at the end of every one* " is not more harmonious in an Irish than in an English dress. The original would be lost beyond recovery, were it not for the reading in B. That lection, though corrupt itself, preserves the requisite word—

A dal faemdeoid iar chach.

Leaving out *iar*, we can restore thus:

A dal femfid cach.	His assembly every one will avoid.

The meaning is, that the sub-chiefs, their advisers and retainers, would not attend his public assemblies when summoned.

The concluding line is very faulty in E and R. The true lection is found in B.

Bid a meinn, ni liach.	He will be in misery, (it is) not a pity.

Before passing from the Irish text, it is of importance to note, that portions of this genealogy in the Book of Ballymote render it feasible to fill up with certainty some of the lacunae in the corresponding part of the lithographed copy of the Book of Leinster.

They are as follow, the matter supplied being enclosed in brackets :—

Book of Leinster, p. 323 *f.*

Line 1, Tri meic [Mechair, idon Cainnech,]
,, 2, Daigri, Gra[nne.]
,, 3, Munech a quo cen[el Baetain.]
,, 4, Da mac Mund[ig : idon, Bel, a quo]
,, 5, Baetan, mac Daigri,
,, 6, Carthach, a quo hUi Mun[dig.]
,, 22, Cendlacan cujus h[eredi-]
,, 23, tas Les Dubain ocus [Cell]
,, 25, Genus autem et hereditas al[iorum defecit].

It is likewise noteworthy, that the compiler in the Leinster MS. gives far less than is contained in the Book of Ballymote.* Moreover, though he inserts the pedigree containing Bel, Munech, Forat and Conla, he does not give the excerpts from the Tripartite. All which goes to prove that the Ballymote lists were compiled by a genealogist dependent upon, or favourably disposed towards, the local families. Such a one, finding testimony so honourable to the ruling septs as that recorded in the Tripartite, would be far too loyal to his patrons to overlook its insertion.

With reference to the Latin text, the main interest lies in the circumstance that it tends to establish the existence of a copy of the Tripartite different from the transcripts which E and R were made from. Compare

* Note B (*b*).

the accounts respecting Carthach : note furthermore the omission in B of the prophecy referring to Cendlachan. The lacunae of B can be filled up from E. Respecting the latter text, as the *Triadis Thaumaturgae* is so scarce, I append the version given therein.* Here it may be well to correct the chief erroneous readings. Most of them were found, we are informed, in the editor's MSS. The rectification is the more necessary, as the lections, with exception of *in ream morti*, have been reproduced in the Rolls' edition of the Tripartite.

For *excusavit causam* (he pleaded the excuse), an instance of a cognate object, we have *excusavit se causâ.* After *debendorum, utrum illi ab aliquo, seu alicui ab eo, nescio* is inserted. The *utrum-seu* clauses are, of course, explanatory of *debendorum*—whether they were due to him from the other, or to the other from him. For *reum morti* we read *in ream morti*, of which nothing can be made; and Colgan places debet after *reddere.* But, in the first place, to correspond with *evadet* it should be *debebit.* The addition, moreover, is needless. *Evadet reddere* is a literal rendering of the native expression *to avoid to give* (= giving); the infinitive being a verbal substantive in Irish. In the next sentence, after *expectaret alumnum suum*, Colgan inserts *volens videre.* The emendation, like the foregoing, could hardly be less felicitous. For there is no lacuna. The subject of the dependent clause is made the object of the principal verb. Similarly, in the Book of Armagh, fol. 11 *a*: Necessitas poscit illos ut pervenirent = Necessitas poscit ut illi pervenirent. (*Poscit* is the historical present.) What is most inexplicable, however, is the failure to perceive in the final paragraph that *hoc* and the duplication of *quod* void the respective sentences of meaning.

The foregoing will contribute to reconcile original workers to the rarity of the *Triadis Thaumaturgae* and the *Acta Sanctorum Hiberniæ.*

The date of the interpolation is fixed approximately by the fact that the passages form portion of the known Patrician materials that were not inserted (perhaps not having been put into shape at the time of transcription) in the Book of Armagh. Very interesting and conclusive evidence is afforded by the headings at folio 19 *b* of that Codex. They become quite intelligible when taken in connexion with the text of the Tripartite.

* Note C (*b*).

Book of Armagh, fol. 19*b*.

(*Each line corresponds with the original.*)

Fuirg, Muindech, Mechar—filii Fora[i]t,
Maic Conli: Muscan, Cellachan, Imchad, Dubthach, Gartne, Lamnid,
Trian, Carthach, Nial, Naindid, Macnise, Conan, Sepis;
debita; alumnus.

Under *Cellachan* and above *debita* are placed marks of correspondence, to denote that it was Cellachan pleaded the dispute regarding *dues*, to excuse his tardy arrival to Patrick. Similarly, *Conan* and *sepis* are respectively dotted overhead, to signify that Conan alleged the laying out of a *hedge*, in explanation of his late coming. *Carthach* and *alumnus*, though not marked, are to be taken in conjunction. They refer to the passage wherein Carthach is represented as requiring to consult his *foster-father* before embracing the faith.

As the Book of Armagh was copied in, or before, A.D. 807, this portion of the Tripartite must have been drawn up in the eighth century: or, at any rate, the materials had become known to the maker of the index during the first decade of the ninth.

That the form (Irish and Latin) in which the incidents appear in the Tripartite (E and R) Recension was the work of a compiler unacquainted with local history, the following appears to establish:—

I. The second quatrain says that Patrick gave a lasting blessing to Mechar. But the Books of Leinster and Ballymote, both independent and unimpeachable authorities, state that the seed of Mechar did not survive.* The former gives but one line from him in six generations. This would date the extinction about the middle of the sixth century. So that the prophecy was falsified a century and a-half before the transcription of the oldest document in which we find it contained.

* *Ni mair a shil*—His seed did not live. Note B.

II. The worst fate was portended for Fuirg. But what is to be found? That his branch of the family, though not so extensive as that of Munnech, was, to say the least, equally distinguished. In 834, Cathal, King of Hy-Many, profaned Clonmacnoise to expel the Munster Vice-Abbot, whom he cast into the Shannon and drowned. This Munster Superior was Flann, son of Flaithbertach, of the Ui-Forgo (*nom.* Fuirg, *gen.* Forgo). Again, in 990, Domnall, son of Lorcan, King of Muscraidhe Tire and Ui-Forgo, with a great multitude, was slain in battle by Malachy, King of Ireland.* The relationship of Flann and Domnall is set forth at foot.† No southern writer, it will be conceded, would have written disparagingly of this sept in the ninth and tenth centuries.

III. With reference to the names, it is worthy of observation that those in the Book of Armagh differ from those in the Books of Leinster and Ballymote. Muscan is certainly the Bel of the genealogies.‡ A mistake in such matters in independent local authorities was very unlikely. Again, Trian and Macnise of the northern list find no counterparts in the southern MSS. In these the appellatives are Raibne and Ailche. That the Irish genealogies are correct here also, is shown by the fact that we find the issue of both Raibne and Ailche identified and the tribal territories designated respectively.§

* Note D.

†

Fuirg, *circa* 450,
Eponymous head of Ui-Forgo.
|
Flaithbertach,
8th in descent from Fuirg.
|

Flann,† 834. (L. B., p. 143 *d.*)	Ogan.
	Mailcainnig.
	Corcran.
	Lorcan.
	Domnall,† 990. (L. B., p. 143 *c.*)

‡ It is strange that, with Muscan and Bel under his hand, the Ballymote scholiast made no attempt to reconcile the discrepancy.

§ Note B.

IV. Finally, the omission of the prophetic passage respecting Cendlachan (incorrectly given as Cellachan in the Tripartite and Book of Armagh) forms authentic proof either that the Ballymote compiler had before him a less diffuse copy of the Tripartite, or that he passed over the portion in question, finding it at variance with the family history.

In conclusion, attention may be directed to two matters of interest in connexion with the literature of the subject.

With reference to the islands in a bog (insolae in gronna), it is matter of regret that the passage containing it has not been given in the Patrician Documents of the Armagh MS. It would thus be, probably, the oldest reference in writing to these habitations.

The Tract in the Book of Ballymote makes another and very important addition to the "lost books of Erinn." At p. 142f, a statement is confirmed as follows:—*Ut invenietur in libar Sochar Lothra*—"as will be found in the Book of Free-Contracts of Lothra" (St. Ruadhan's monastery, in Lower Ormond). Farther down, of another fact it is asserted:—*Ata im* (*in*) *eodem libro, paulo post.*—"It exists in the same book, a little after." The value of the circumstance that a Book of Muniments lay accessible for reference in a public establishment will be readily conceded.

[Note A.

NOTE A.

The Rolls' Edition of the Tripartite Life of St. Patrick.

The Tripartite has been recently issued in the series of Chronicles and Memorials published under the direction of the Master of the Rolls of England.* Accuracy and completeness of treatment might not unreasonably be looked for in such a work. It is disappointing to find them both absent, as the following will show :—

(*a*) *Recension.*

With reference to the two MSS. that supply the text, no attempt has been made to determine whether they are identical or different in origin. We have instead a catalogue of one-and-thirty pages, describing the contents of Rawlinson, B 512. The portion relevant is contained in ten lines. In this decade, the title and closing words of the Codex present four variants, by comparison with the forms printed at pp. 2, 264, respectively.

A remarkable hiatus occurs in the MS. Folio 6 ends with *sle-*, the first syllable of *Slebe miss* (Slemish, Co. Antrim). Folio 7 begins with *-is*, the final syllable of *bathis* (baptism), a word occurring in the Irish Hymn of St. Patrick. Hereupon, three queries of radical import immediately suggest themselves. Has the vacuum been caused by excision in the present MS.? Or, did it exist in the exemplar? If so, did the scribe copy on, in ignorance of the break in the continuity? To the editor, however, the lacuna was a lacuna and nothing more. Similarly, the linguistic standard of the Codices has been estimated neither independently nor by comparison.

The illiteracy of transcription so conspicuous, especially in R, has been passed over in silence. Nay more, the editor has declared and repeated, that the copyist of the Bodleian MS. was "a careful scribe, and learned in his own tongue." Accordingly, the bilingual sentence quoted above† is accepted as correct in the main. To complete its accuracy, the contraction *epis.* (*i. e.* episcopis) is lengthened into *episcoporum*, to match *docum* with the proper case. The native heading given thereafter‡ is received as faultless. *Menstir*, we are informed (p. clxxxvii.), signifies a credence-table, which was always probably of metal. It comes, the Index states, "from Low-Latin menisterium = ministerium, abacus, mensa in qua pocula reponuntur.—Du-Cange." The editor, namely, having lighted upon the first *ministerium* in Du-Cange, copied the meaning.

* *The Tripartite Life of Patrick, with other Documents relating to that Saint. Edited, with Translations and Indexes, by Whitley Stokes, D.C.L., LL.D.*, &c., 1887.

† *Supra*, p. 185.

‡ *Loc. cit.*

The result is that he makes Irish bishops of the olden time journey upon duty with metal tables strapped to their backs. *Lege totum* is plainly not his motto; else he would have found that the *ministerium* placed second in the *Glossary* is the vocable required.*

The text, with the exception of a few pages, has been selected from R. The variants, however, show at once that the transcription of E is better in form and substance. Thus, to give an example of each language, we have (p. 96), *Bite, filium Assici* overhead, and the E reading, *Bite, filium fratris Assicus* [*-i*], at foot. Elsewhere (p. 148), we find *Bite, filius fratris Assici,* from the Tripartite, and (p. 313) *Betheum, filium fratris Assici* from the Book of Armagh (fol. 11, c, d). No explanation is hazarded by the editor.

On the same page (96), we read: *Dindail tuargabad isin tiprait . . . ocus ita for bruach.* Here *isin* (in the) and *ita* (wherein is) are voces nihili; *asin* (out of the) and *ata* (which is) being the forms requisite. "From the rock which was raised *out of the* well . . . and which is *on the* brink." The correct readings, as preserved in B, have been stowed away in a footnote. Yet the English version is made from them.

The same sentence (explanatory of the name Elphin) is a typical example of the redaction. (I omit portions not pertinent to the present purpose.) "Dindail . . . nominatur locus *Ail-find;* de aqua nuncupatur—The place is named *Ailfind* from the stone (*ail*) . . . it is called from the water [*find*(fair)]." But the text lacks nothing. Dindail . . . nominatur locus *Ail; find* de aqua nuncupatur—From the rock . . . the place is named *Ail; find* it is called from the water.

(*b*) *Transcription.*

The *facsimile* prefixed affords an unerring test of the accuracy with which the text has been transliterated. I subjoin tabulated results:—

ROLLS' EDITION.	RAWLINSON, B 512.
P. 192, l. 21, Slebtiu.	Fol. 23*a*, l. 3, Sleibtiu.
194, l. 1, ifirn.	„ 12, iforn (*lege* ifirn).
„ 4, forothaigestar.	„ 16, forothaigester.
„ 9, atrubairt.	„ 23, atrubert.
„ 13, foirenn.	„ 27–8, forenn.
„ „ munntir.	„ 28, munnter.
196, l. 12, maicc (g. sg.).	23*b*, gl. l. 24, mac (g. pl.).

Apart from this list, there is evidence that the contractions were both misread and

* *Glossarium,* &c., Paris, 1733, col. 784–5. Ed. Henschel, p. 419–420.

read at random. We have, for instance (pp. 184–5), "*for a cenn*—on their heads." The MS. has a contraction marked over the first *n*. That it was not intended for a single *n*, is shown in two ways. First, *cenn* is genitive, and cannot consequently be governed by *for;* which requires a dative or accusative. Secondly, folio 24 *c* (p. 206) gives *cenn* twice, each time written in full. The reading, accordingly, is *cendaib* (the dative upon *for*), as on folio 12 *c* (p. 102), where no compendium is employed. At folio 19 *b* (p. 164), *Coelbad* (gen.) occurs uncontracted. In two other places the word is in shorthand. The first (19 *b*, p. 164) the editor reads *Coelbad;* the second (19 *d*, p. 166), *Coelbaid.* One instance is too whimsical to be overlooked. "Sadid eclasc ann ropu dos iarum—He therein sets a horse-rod, which afterwards became a bush" (pp. 158–9). The absurdity dawned upon the editor in revision. Accordingly, we have (p. 692) "*for* eclasc *read* eclais;" "*for* horse-rod *read* (wattles for) a church." A church changed into a bush is a novel instance of thaumaturgy.

(*c*) *Translation.*

To show the perfunctory manner in which the English version has been carried out, the Latin is unrepresented in ten places, and the Irish in more than forty. As regards the fidelity, nine-tenths of the work had the advantage of a published translation. Notwithstanding, two lists of corrigenda have had to be appended. The first contains eighty, the second twenty-six, items. The field was unexhausted still. My search resulted in an addition of double the previous totals.

Most of the errors arise from lack of the intuitive perception which comes exclusively from a knowledge of the living tongue. Take, for example, *do rala ind aimsir geimred.* The native speaker at once sees that this means *it happened in the winter time.* But here (p. 12) the "bookish theorick" makes it *the winter time came.* A much more important instance is the following (p. 4) :—*Is he . leth ataebi ind aisneissi.* The editor's acquaintance with *leth ataebi* is not of recent date. As long ago as 1877 he rendered it by "parallel passage,"* and explained "leth a-toibe; lit. *half of its side.*" † It is unnecessary to disprove *toibe* as genitive of *toeb* by pointing to *tuib* in the Stowe Missal (fol. 64 *a*) and *taib* in Lebar Brec (p. 4 *a*). For the author has abandoned the position. He now takes up new ground: "pres. ind. sg. 3, leth ataebi = leth atoibe, the part that adheres, abuts, the context" (p. 641). In accordance with one of the primary corrections (p. 674), the passage means "this is the context of this declaration."

Part that adheres = context is an equation that needs no refutation. Equally devoid of foundation is the Irish. First, this place and an emendation [*is cell* [*and*]—

* *Three Middle-Irish Homilies*, p. 2. † *Ib.*, p. 135.

"there is a church [there]" (p. 420–1)] show that the editor has not mastered the radical fact that in Irish *is* constitutes a copula, and can, accordingly, never predicate. Moreover, were *ataebi* indicative, the antecedent, *leth*, would take the article. Compare *alleth* (= *an leth*) *crin irabi*—"the withered half in which he was" (pp. 57–8). Thirdly, *aisneis* is not the genitive, but the nominative. The genitive form, *aissnesen* (Old Irish *aisndisen*), appears elsewhere (p. 256). The correct lection is, consequently, *Is he . . . leth atoib[th]e ind aisneissi* [*codu*, &c.]—"This narrative [at the place (Isaiah ix. 1), &c.] is the context [lit. *side of adherence*] (of v. 2)." The meaning is that the second verse fits in after the first—an explanation which it does not require any special critical sagacity to discover.

Statements and rectifications in similar detail respecting redaction and translation would occupy many pages. I shall, therefore, conclude this portion with specimens of the manner in which texts specially facile of treatment have been dealt with.

Tripartite, Rolls' Edition, p. 64.	Egerton MS., fol 4.
"Euntes ergo nunc," id est, dum mea potestas in omni terra et non in Iudea tantum invenitur. Quod dudum pro[h]ibui dicendo, "In viam gentium ne abieritis," nunc vobis concedo et praecipio, "Ite, docete."	*Euntes ergo nunc* [Matt. xxviii. 19]: id est, dum mea potestas in omni terra et non in Iudea tantum invenitur, quod dudum[1] prohibui, dicendo: *In viam gentium ne abieritis* [Matt. x. 5], nunc vobis concedo et precipio.
Aptus ordo[1] doctrina ante bautismum. Non enim potest fieri ut corpus babtismi recipiat sacramentum nisi ante[quam] anima fidei suscepit veritatem.	*Ite, docete.* Aptus ordo,—doctrina ante baptismum. Non enim potest fieri, ut corpus baptismi[2] recipiat sacramentum, nisi ante anima fidei susceperit veritatem.[3]
[1] [Figure [1] is placed after *ordo*, but no note has been appended.]	[1] MS. du di. [2] MS. babtismi. [3] Primum docent omnes gentes, deinde doctas intingunt aqua. Non enim, &c. S. Hier. in Matt., Lib. iv., cap. ult.

Here the meaning has been totally missed. For the first paragraph forms a comment upon *nunc*, and should not have been divided into two sentences. The object of *praecipio* is not *ite, docete*, but the antecedent of *quod*. The next paragraph explains *ite, docete*, which should be prefixed thereto accordingly. The final passage the editor, of course, did not know was taken from St. Jerome. The insertion of [*quam*], whereby the sense is altered diametrically, is due to the lection of the *Triadis Thaumaturgae* (p. 117). But Colgan, who had something more than a school-boy's knowledge of Latin, has *ne* in the antecedent clause.

TRIPARTITE, ROLLS' EDITION, p. 172.

Mirabilis Deus in sanctis suis, *i.e.* per signa laudabilis; quae sive in angelis suis sive in hominibus sanctis qui eius iusionibus obsecundant operatur ipse. Est enim qui facit mirabilia magna solus, unde Iacobus Apostolus dicit: Omne datum optimum et omne donum perfectum, desursum est, discendens a Patre luminum. At vero tunc mirabilis Deus in sanctis suis in conspectu gentium revelatur quando eis regna celorum pollicentibus (.i. doctoribus) dedit potestatem infirmos curandi, mortuos suscitandi, leprosos mundandi,[5] demones ieciendi, cecos illuminandi, claudos et sordos sanandi, et cetera. Huiuscemodi vero[6] magnitudinem promissorum probat magnitudo signorum.

[5] Sic E.; sanandi, R.

[6] Sic E.; ut, R. [The contraction here misread *vero* occurs in *sicut* (*infra*, p. 203) and elsewhere in E.]

EGERTON MS., fol. 11.

Mirabilis Deus in sanctis suis [Ps. lxvii. 36]: id est, per signa laudabilis, quae seu in angelis suis, seu in hominibus sanctis qui eius iussionibus[1] obsecundant, operatur. Ipse est enim *qui facit mirabilia magna solus* [Ps. cxxxv. 4]. Unde Iacobus Apostolus dicit: *Omne datum optimum*[2] *et omne donum perfectum desursum est, descendens*[3] *a Patre luminum*[4] [i. 17]. At vero ideo[5] mirabilis Deus in sanctis suis in conspectu gentium revelatur, quando eis regna coelorum pollicentibus[6] dedit potestatem infirmos curandi, mortuos suscitandi, leprosos mundandi, demones iaciendi,[7] caecos illuminandi, claudos et surdos[8] sanandi et caetera huiuscemodi, ut magnitudinem promissorum probet magnitudo signorum.

MS. READINGS

[1] S singled. [2] Optimi. [3] Discendens. [4] Lumini. [5] Vel tunc is placed overhead, to signify that *tunc* is an alternative reading. [6] Id est, doctoribus is given as a gloss on pollicentibus. [7] Demonis ieciendi. [8] Sordos.

In the left-hand column, *ipse* belongs, of course, to the sentence following. The passage beginning with *at* has been entirely misunderstood. *Tunc* is an alternative lection in E. It refers to *quando*. But the construction is: *ideo* mirabilis revelatur *ut* magnitudo probet. *Ieciendi* as the gerund of *iacere*, *to cast out*, deserves notice.

Omne datum optimum is Englished *every good*. *Eis*, which relates to *gentium*, and is the dative upon *pollicentibus*, has not been translated. Finally, *vero* is taken to mean *so that*.

(*d*) *Illustrative Material.*

Respecting textual illustration, no effort has been made to supply linguistic or historical notes. In lieu, perhaps, we get a diffuse introduction. Its value upon important points can be readily tested. The grammatical forms of both MSS., all corrupt

as we have seen them to be, are marshalled through five-and-twenty pages, to prove that the Tripartite was composed in the eleventh century. Under nouns, *pecaid*, a scribal corruption for *pectho* (peccati), is accepted as "early middle-Irish." Under adjectives, *deniu*, a grouping of three words, is given as comparative of *dian*, swift. Under pronouns, infixed *n* (= pl. 3) does not appear, though it occurs in the work no less than three times (*rolliuc* = *ro-n-*[*sh*]*luic*, p. 92; *rollasat* = *ro-n-lasat*, p. 142; *ronnig* = *ro-n-nig*, p. 144). Under verbs, rather than admit *n* = pl. 3, the last example is divided into *ro-nnig*, and classed a reduplicated preterite. But unless *fonenaig*, given (p. 420) from the Book of Hymns, is all wrong, the reduplicate of *ronnig*, it is not difficult to see, must be *ronenig*.

In literature, the following is quoted from a German source: "In an old fragment of Scotic Liturgy Patrick is named as Patron of Ireland" (p. cxvi.). This is the well-known St. Gall Fragment, printed and annotated in the Academy's *Transactions* (Pol. Lit. and Antiq., xxvii. 233–7). The inaccuracy of the statement respecting St. Patrick can be seen by reference to p. 234. Similarly, with verified data available, the editor assumes (p. clxxxv.) that the Stowe Missal contains sixty-six folios. The baptismal rubric, *Discendit*, &c., he quotes from fol. 56 *b*, and gives the reading *aspergitur* (p. clxxxiii.).

In liturgy, perhaps to show the vitality of error, notwithstanding the extracts and references given in these *Transactions* (*ubi sup.*, p. 157), the Cursus, or Office, is equated with the Mass (p. cxix.). The "hallowed bread broken up for the Eucharist" is, we are told (p. clxxxv.), called *eylogia* by Adamnan. The expression (Vit. Col. ii. 13) is *in refectorio eylogiam frangere*. But Dr. Reeves, who knew his subject, did not thus annotate. The editor leaves it to be inferred that he consulted Gregory of Tours. Yet he failed to learn the meanings of *eulogiae* from the eleven passages in which the word is employed by the Saint.*

One cannot more appropriately conclude than by recording a fresh triumph of the inductive method, this time in the domain of Divinity. "Polygamy existed, and hence Patrick, like St. Paul, requires for the bishopric of Leinster a husband of one wife" (p. clxviii.).

* Hist. Franc. iv. 35; v. 14, 32; vii. 1; viii. 2, 20; x. 16. Glor. Conf. xxxi. Vit. PP. viii. 3; xx. 4. Mirac. S. Mart. iv. 21.

Menard did not fully utilize these references in his Note upon *eulogias* of the *Concordia Regularum* (c. lxi.). Hist. Franc. v. 14 and viii. 20 are the only quotations given.

It may not be out of place to observe that Adamnan's (V. C. ii. 7) *eylogia* = *benedictio* explains the passage in the Book of Armagh (fol. 6d): portentur nobis beneficia eius. In the former case, the material was salt; in the latter, water.

NOTE B.

(*a*) Extract from Book of Ballymote, p. 141 *b*.

De ghenelacht Muscraigi Tiri. . . .

Tri meic Forait, maic Conla, idon, Fuirg (141 *c*) (a quo hUi Forga*); Meachar (Ocus ni mair a shil. Tri meic Mechair, idon, Cainnech, Daig[ri], Granne.); Muindech (a quo Cenel mBaeta[i]n). De quibus Patricius dixit, quando baptizavit et benedixit eos.

I.

Credis Muindech mar
Do Padraig ria cach.
Robe fo rath uaith
Rigi uad go brath.

II.

Credis Meachar cerb,
Ba fer con deigfir;
Dobert bennacht mbuain—
Cetlud dó fri rig.

III.

Frithbert in fear fergach, Fuirg,
Gerbo riglaech liath;
A dal faemdeoid iar chach,
Bid ameinn, ni liach.

Da mac deg Muindig, idon, Bel (a quo Baetan, mac Daigri, maic Beil); Carrthach (a quo hUi Muindig Chroich); Nainded (a quo hUi Naindedha filed an Ib Muindig ig Croich); Raibne (a quo hUi Raibne Chilli Cere ocus hUi Raibne fri sliab anes); Ailche (a quo hUi Lachtnain o Ard na nEn ocus hUi Luinin Droma Inbir);† Imchad (a quo

* Ferga, MS.

Of the genealogy of Muscraige Tire. . . .

[There were] three sons of Forat, son of Conla, namely, Fuirg (from whom Hy Forgo), Mechar (and his seed did not live. Three sons of Mechar, namely, Cainnech, Daigri, Granne.), Muindech (from whom Cenel Baetain). Of whom Patrick said, when he baptized and blessed them:

I.

Muindech the great believes
In Patrick before every one.
There shall be under singular felicity
Kingship from him to doom.

II.

Meachar the vigorous believes,
He was a man of real sincerity;
He gave a lasting blessing—
Companionship for him with a king.

III.

The angry man, Fuirg, resisted,
Though he was a royal hero hoary;
His assembly every one will avoid,
He will be in misery, (it is) not a pity.

[There were] twelve sons of Muindech, namely Bel (from whom Baetan, son of Daigri, son of Bel), Carthach (from whom Hy Muindig of Croch), Nainded (from whom Hy Naindedha, who are in Hy Muindig at Croch), Raibne (from whom Hy Raibne of Cell Cere [Kilkeare] and Hy Raibne, towards the mountain to the

† Macbir, MS.

Conai[n]g, mac Brocain, fil in Ath Cleithe ocus Coluim mac Bracain, fil i Toim in Baird, ocus Coman, mac Bracain, fil i Cill hUi Maillachtna); Cendlechan (cujus hereditas Leis Dubain ocus Cell Cnamraige). Genus autem et hereditas aliorum nunc defecit, id est, [141 *d*] Niall, Dubtach, Gartna, Lamnaid, Conaind. Qui excusavit Patricio* causa[m] sepis ponendae. Cui Patricius dixit: Semen tuum nec muris nec sepibus poterit habitaculum vel agros integre munire: nam si [terram] fodiunt, cito cadet; si sepem [ponunt], cito cadet; si insolas in gronna, non pos[s]unt stare.

Bel prius venit ad Patricium, et baptizavit eum. Et ideo benedixit eum, et dedit semini ejus regnum semper.

Carthach cito credidis[s]et, si tantum s[c]iret consilium alumni† sui. Patricius dixit: Viri prudentes et ingeniosi mundialibus rebus ex semine tuo erunt.

south), Ailche (from whom Hy Lachtnain, from Ard na nEn, and Hy Luinin of Drom Inbir [Drumineer], Imchad (from whom Conaing, son of Braccan, who is in Ath Cleithe, and Colum, son of Braccan, who is in Toim in Baird, and Coman, son of Braccan, who is in Cell Hy Maillachtna), Cendlachan (whose heritage is Less Dubain and Cell Cnamraige). The lineage as well as the heritage of the other [sons] hath now failed.

(*b*) Extract from Book of Leinster, p, 323, *f.*

Genelacht Múscraige Tire. . . .

Trí meic Forait, maic Conlai; idon, Fuirg (a quo hUi Fhargga); Mechar (a quo hUi Conmail . . . ocus ní mair a s[h]íl. Trí meic [Mechair, idon Cainnech,] Daigri, Gra[nne.]; Munech (a quo Cen[el mBaetain]. Da mac deg Mund[ig, idon, Bel (a quo] Baetan, mac Daigri, [maic Beil)]; Carthach (a quo hUi Mun[dig]); Nannid (a quo hUi Nannida filet in hUib Mundig); Raibni (a quo hUi Raibne Cilli Cére, ocus hUi Ra[i]bni fri sliab aness);

Genealogy of Muscraige Tire.

[There were] three sons of Forat, son of Conla, namely, Fuirg (from whom Hy Forga), Mechar (from whom Hy Conmail. . . . And his seed did not live. Three sons of Mechar, namely, Cainnech, Daigri, Granne.), Munech (from whom Cenel Baetain). Twelve sons of Munech, namely, Bel (from whom Baetan, son of Daigri, son of Bel), Carthach (from whom Hy Mundig), Nannid (from whom Hy Nannida, who are in Hy Mundig), Raibni

* Exduxit Patricium, MS.

† Alumpni, MS.

Ailchi (a quo hUi Lachtnáin o Ard na nEn ocus hUi Lunín Dromma Inbir, ocus Cuanu; mac Cailche—Unde Glais Chuana.); Imchad (a quo[1] Conaing, mac Braccain, fil in Ath cleithi ocus Colum,[2] mac Braccain, fil i Tóm in Baird, ocus Comman,[2] mac Braccain, fil i Cill hUi Maellachtna); Cendlachan (cujus h[eredi]tas Les Duba[i]n ocus [Cell] Chnamraide ocus Conach). . . . Genus autem et hereditas al[iorum defecit.]

[1] The MS. has hUi—Hy Conaing. But the meaning is that the three sons of Braccan were saints. Their names and places of establishment are respectively given.

[2] Lonan is given as an alternative reading over Colum and Comman.

(from whom Hy Raibne of Cell Cere, and Hy Raibni towards the mountain to the south), Ailchi (from whom Hy Lachtnain from Ard na nEn, and Hy Lunin of Drom Inbir, and Cuanu, son of Cailche—whence Glais Chuana), Imchad (from whom Conaing, son of Braccan, who is in Ath Cleithe, and Colum, son of Braccan, who is in Tuam in Baird, and Comman, son of Braccan, who is in Celle Hy Maellachtna), Cendlachan (whose heritage [is] Les Dubain and Cell Chnamraide and Conach). . . .

The lineage as well as the heritage of the other [sons] hath failed.

NOTE C.

(*a*) Extract from the Tripartite Life, Part III.

Egerton MS. (*British Museum*).

Luid Patraic im Muscraidi Tiri babtizare atque predicare fidem et fundare fidem. Ibi inveni[un]tur tres fratres, illius regionis potentes—Furic et Muinnech et Mechar: meic Fora[i]t, maic Conla. Credit* Muinech protinus. Ocus rombaist Patraic, ocus rombendach, ocus foraccaib oirdnidi laech ocus clerech uad co brath, ocus ardrigi a thiri uad co brath. Sicut dixit, conoterae. [?]

Patrick went into Muscraide Tire to baptize, and to preach the faith, and to establish the faith. There are found three brothers, powerful men of that region—Furic and Muinnech and Mechar: sons of Forat, son of Conla. Muinech believes straightway. And Patrick baptised him, and blessed him, and left distinguished laics and clerics from him to doom, and the chief-kingship of his territory from him to doom. As . . . said.

* Creidit, MS.

I.

Creitis Muinnech mar
Do Patraic ria cach ;
Combia for a thuaith
Toisech uad co brath.

I.

Muinnech the great believes
In Patrick before everyone ;
So that there will be over his territory
A chief from him to doom.

II.

Creitis Mechar cer[p],*
Pa fer condilc fir ;
Dobert Patraic bendacht mbuain—
Cetlad do fri rig.

II.

Mechar the vigorous believes,
He was a man of true judgment ;
Patrick gave a lasting blessing—
Companionship for him with a king.

III.

Frithmbert infer fercach, Fuirgg,
Ciar bu rigla[e]ch liath ;
Adal fu diud iar cach,
Bith a min co brath, ni liach.

III.

The angry man, Fuirgg, opposed,
Though he was a royal hero, hoary ;
His assembly at the end after every one,
To be in misery to doom, (is) not a pity.

Sicut prediximus, Muinnech a fratribus in regnum segregavit. Duodecim vero Muindech filios sustinuit ad se venire : hoc est, Muscan, Cellachan, Imchath, Dubthach, Gartne, Lamnid, Trian, Carthach, Niall, Naindid, Macnise, Coninn. Qui tarde venerunt, excepto Muscano ; cui per hoc pre omnibus fratribus regnum destinavit. Quod adhuc manet sine commotatione.

Coninn excusavit causam sepis exponendae. Cui Patricius dixit, quod progenies ejus nunquam in æternum muris aut sepibus potuisset habitacula aut agros ad integrum munire. Nam si terram fodiunt, dehiscit ; si sepem ponunt, cadit cito ; si insolas in gronna (ponunt), nunquam firmiter possunt stare.

Cellachan dixit quod causa munerum debendorum, utrum illi ab aliquo seu alicui ab eo, tarde pervenit. Cui Patricius dixit : omni spatio quo apud Muminenses amnestia mea vitiata fuerit, et tu transgressus fueris, etiam si alii liberi fuerint aliquâ causâ, nunquam tu et gens tua evadet aut reum morti aut septem ancillas† reddere.

Carthach dixit quod credidisset, si tantum expectaret alumnum‡ suum, utrum prohiberet eum an non. Patricius dixit, autem§ [quod] prudentes et ingeniosi mundialibus causis ex progenie ejus fuissent, regno alienati. Sicque verbum unicuique ex eis dixit quod impletum est.

* See *supra*, p. 187, note †.

† Ancellas, MS.

‡ *A aité, his fosterer,* is placed overhead as a gloss upon alumnum suum.

§ Hautem, MS.

(*b*) TRIADIS THAUMATURGAE, &c., p. 159.

[The portion in Roman type is Colgan's version of the Irish; the italicized sections are, he says (p. 169), the old Author's Latin.]

Desiis relictis, progressumque in fide disseminanda continuans, venit Sanctus Patriarcha in Muscragiorum alteram quae vulgo Muscraighe Thire appellatur: per cuius fines progrediens populumque in salutari doctrina instruendo et baptizando, offendit tres regionis illius viros potentes, Furecum, Muinechum et Meacharium Foradio, Conlae filio, natos. Muinechus autem annunciatam Christi fidem salutaremque doctrinam statim amplexus et a sancto antistite sacra salutaris lavacri unda intinctus est: qui et devotum filium recens regeneratum sua benedictione ditavit, predicens quod ex eius progenie non solum multi preclari belli duces multique conspicui antistites, sed et supremi terrae illius sint proditurí principes. Credidit et tardius tertius fratrum, de cuius posteritate predixit Patricius quod essent horum emuli et principibus inobedientes. Furechus, vero, fratrum aetate primus, sed et re et nomine postremus, vir ferocis animi duraeque et inflectibilis cervicis, nullo modo flecti potuit ut annunciatam salutis viam sequeretur, sed in sua caecitate miser, minime miserandus permansit usque in finem.

Patricius igitur, *ut prediximus, Munechum a fratribus in regnum segregavit. Duodecim vero Munechi filios sustenuit ad se venire—Muscan, Keallachan, Imchadh, Dubthach, Gartne, Lamnith, Trian, Carthach, Niell, Nandith, Mac nesse et Conenn. Qui tarde venerunt, excepto Muscano, cui propter hoc pre omnibus fratribus regnum destinavit vir Dei: quod adhuc manet sine commutatione.*

Conenn vero excusavit se causâ sepis exponendae. Cui Patricius dixit quod progenies eius non in aeternum muris aut sepibus potuisset habitacula aut agros ad integrum munire: nam si terram fodiunt et dein sepem ponunt, cadit cito; si insolas in gronda, nunquam firmiter possunt stare.

Keallachan dixit quod causa munerum debendorum (utrum illi ab aliquo, seu alicui ab eo, nescio), tarde pervenit. Cui Patricius: Omni spatio quo apud Muminienses amnestia mea vitiata fuerit, et tu transgressus fueris, etiamsi alii liberi fuerint aliquâ causâ, nunquam tu et gens tua evadet in ream morti aut septem ancillas reddere debet.

Carthach dixit quod credidisset, si tantum expectarent alumnum suum, volens videre *utrum prohiberet eum an non. Patricius dixit quod prudentes et ingeniosi mundialibus causis ex se et progenie eius fuissent regno hoc alienati. Sic quod verbum unicuique ex eis dixit: quod impletum est.*

NOTE D.

Extracts from Irish Annals respecting Hy Forgo.

(*a*) *Chronicon Scotorum.*

Saruccadh Cluana muc Nois do Cathal, mac Aililla, Ri hUi Maine, for secnabad Muman, idon, Flann, mac Flaithbertaigh, do Uib Forga, contard isin Sinainn, contorcair. Dliged vii. cell [cumal] ind.

—[Marginal], A.D. 823.

Profanation of Clonmacnoise by Cathal, son of Ailill, king of Hy Many, against the Munster Vice-abbot, namely, Flann, son of Flaithbertach, of Hy Forga, so that he cast him into the Shannon, and he perished. Judgment of seven churches [bondmaids] therein.

(*b*) *Annals of Four Masters.*

Saruchad Cluana mic Nois do Cathal, mac Ailella, tigherna Ua Maine, for Fhlann, mac Flaithbhertaigh, d'Uibh Forggo, prioir a Mumhain, condotard isin Sionainn, con dorchair. Dlighedh vii. ceall [cumal] do Chiaran ocus mainchine mor.

—A. D. 834.

Profanation of Clonmacnoise by Cathal, son of Ailill, lord of Hy Many, against Flann, son of Flaithbhertach of Hy Forgo,* Prior from Munster, so that he cast him into the Shannon and he perished. Judgment of seven churches [bondmaids] to [St.] Ciaran [founder of Clonmacnoise], and much [other] requital.

Respecting *dliged vii. cell*, O'Donovan † explains that the revenues of seven churches were forfeited to Clonmacnoise. That were vicarious punishment indeed. For there is no evidence that any ecclesiastic aided or abetted the profanation. Hennessy ‡ says that Cathal was adjudged to endow seven churches. But, amongst Old-Irish canonical penalties you will search in vain, if I mistake not, for church endowment, or sequestration of same. The mulct of seven bondmaids (*secht cumala, septem ancillae* = twenty-one milch-cows) is, however, well known to have attached to offences of a kind with those perpetrated by the King of Hy Many.§

Herein lies the clue to the meaning. Tigernach's original was: *Dliged secht cumal and*—Judgment of seven bondmaids therein. This he Latinized: *Judicium vii. ancillarum in eo.* By the native interchange of *e* and *i*, *ancillarum* was written

* "A tribe seated at and around Ardcroney, near Nenagh, Co. Tipperary."—O'Donovan, *F. M.* i. 451.

† *Loc cit.*

‡ *Chron. Scot.*, p. 132.

§ *Patricius dicit:* Omnis qui ausus fuerit ea quae sunt regis, vel episcopi, aut scribae furari,

ancellarum. Then some scholiast, of the same size of learning as Mac Firbis and the O'Clerys, unknowing the technical sense of the vocable, read it *cellarum*, and retranslated the entry: Dliged secht *cell* ind—Judgment of seven *churches* therein.

(*c*) *Annals of Tigernach.*

[A.D. 990] Cath fordromma la Mael-[sechnaill], du atorcair Dom[nall], [mac] Lorcain, [rí] Muscracide [*sic*] tiri ocus [hUi] Forga; ocus ar mor arcena.
—O'Conor, *Rer. Hib. Script.*, ii. 265.

Prælium Fordromense gestum a Maelsechlanno, in quo occisi sunt Donaldus, fil. Lorcani, Rex regionis Muscradiensis [*recte* M. Tiri], et O'Forga, et strages magna aliorum facta [est] similiter.

(*d*) *Annals of Ulster.*

990. Cath Cairn Fordroma ria Maelsechlainn for Tuathmumain, i torchair Domnall, mac Lorcain, ri hUi Forca, et alii multi.

990. Battle of Carn Fordroma, by Maelsechlainn over Thomond, wherein fell Domnall, son of Lorcan, king of Hy Forgo, *et alii multi.*

Fordroma, the reading of Tigernach, is probably correct. The place has not been identified.

(*e*) *Chronicon Scotorum.*

Maidm, no cath, Cairn Fordroma, ubi multi occisi sunt, dú attorcair, Domnaill, mac Lorcain, Rí Muscraidhe Tíre ocus hUi Forga, ocus ar mor arcena, la Maolseclainn, mac Domnaill.
—[Marginal], A. D. 988.

Defeat, or battle, of Carn Fordroma, *ubi multi occisi sunt*, a place wherein fell Domnall, son of Lorcan, king of Muscraide Tire and Hy Forga, and [there was] great destruction besides, by Maelsechlann, son of Domnall.

aut rapere, aut aliquid in eos committere, parvipendens dispicere, vii. ancillarum pretium reddat, aut vii. annos peniteat cum episcopo, vel scriba.—*Synod. Hibern.*, c. 8.

Sinodus Hibernensis ait: Omnis, &c. [Same as foregoing, with *aut* for *vel* and *aut scribae* omitted.]—*Col. Can. Hib.*, xlviii. 5.

Synodus Hibernensis ait: Sanguis episcopi, vel excelsi principis [*i. e.* abbatis], vel scribae, qui ad terram effunditur, si coliiro indiguerit, eum qui effuderit sapientes crucifigi judicant, vel vii. ancillas reddat.—*Synod. Hibern.*, c. 1.

The eric of seven bondmaids is also mentioned in the Tripartite, note C. (*a*), *supra*, p. 203.